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The absence of logarithmic divergences in the spin and charge density correlations of the 1d Hubbard model

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Abstract
We perform a rigorous computation of the spin and charge density correlations of the 1d repulsive Hubbard model at weak coupling, focusing on the properties of the Fourier transform at momentum 0 and $\pm 2p_F$, if p_F is the Fermi momentum. We prove that the interaction changes the singularity at $\pm 2p_F$ (the discontinuity in the derivative becomes a power law singularity) while the singularity at 0 is essentially unchanged. Our results show that the logarithmic divergences at zero momentum recently found in [7], which would be in contrast with Luttinger liquid behaviour, are indeed spurious.

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1. Introduction

In some recent papers, see [5–7], the spin and charge density correlations in the 1d repulsive Hubbard model have been considered from a renormalization group (RG) point of view. In particular, it has been found in [7] that the interaction apparently produces logarithmic divergences at zero momentum (see (15), (16) of [7]), which would be in contrast with the expected Luttinger liquid behaviour of the 1d Hubbard model. Aim of this paper is to rigorously compute the spin and charge density correlations, and to prove that there are indeed no logarithmic divergences at zero momentum in the spin or charge correlations; their appearance in [7] is just an artefact due to the approximations involved in the analysis.

The Hamiltonian of the 1d *Hubbard model* with *non-local* interaction is

$$H = -\frac{1}{2}\sum_{x\in\Lambda,\sigma}\left(a^+_{x,\sigma}a^-_{x+1,\sigma}+a^+_{x+1,\sigma}a^-_{x,\sigma}\right)+U\sum_{\substack{x,y\in\Lambda\\ \sigma,\sigma'}}v(x-y)a^+_{x,\sigma}a^-_{x,\sigma}a^+_{y,\sigma'}a^-_{y,\sigma'} - \mu\sum_{x\in\Lambda,\sigma}a^+_{x,\sigma}a^-_{x,\sigma}, \qquad (1)$$

where Λ is an interval of L points on the one-dimensional lattice of step 1, which will be chosen equal to $(-[L/2],[(L-1)]/2)$ and $a^{\pm}_{x,\sigma}$ is a set of fermionic creation or annihilation operators with spin $\sigma = \pm$ satisfying periodic boundary conditions; $U > 0$ is the coupling, $v(x-y)$ is a short range interaction and μ is the chemical potential.

We consider the operators $a^{\pm}_{\mathbf{x},\sigma} = \mathrm{e}^{Hx_0} a^{\pm}_{x,\sigma}\, \mathrm{e}^{-Hx_0}$, $\mathbf{x} = (x, x_0)$ and x_0 will be called time variable. Many physical properties of the fermionic system at inverse temperature β can be expressed in terms of the *Schwinger functions*, that is the truncated expectations in the grand canonical ensemble of the time-order product of the field $a^{\pm}_{\mathbf{x},\sigma}$ at different spacetime points. If

$$\langle X\rangle_{L,\beta} = \frac{Tr\, \mathrm{e}^{-\beta H} X}{Tr\, \mathrm{e}^{-\beta H}}, \tag{2}$$

the Schwinger functions are defined as, if $\varepsilon = \pm$ and $x_{0,1} \geqslant x_{0,2} \geqslant \cdots \geqslant x_{0,n}$,

$$S_{L,\beta}(\mathbf{x}_1, \varepsilon_1, \sigma_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n) = \left\langle a^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1} \cdots a^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n}\right\rangle_{L,\beta} \tag{3}$$

and $\lim_{L,\beta\to\infty} S_{L,\beta} = S$. The charge and spin density *correlation functions* are given by, if $x_0 \geqslant y_0$,

$$N^{\varepsilon}_{L,\beta}(\mathbf{x}-\mathbf{y}) = \left\langle \rho^{\varepsilon}_{\mathbf{x}} \rho^{\varepsilon}_{\mathbf{y}}\right\rangle - \left\langle \rho^{\varepsilon}_{\mathbf{x}}\right\rangle\left\langle \rho^{\varepsilon}_{\mathbf{y}}\right\rangle, \tag{4}$$

where $\varepsilon = 0, 1$, $\rho^0_{\mathbf{x}} = \frac{1}{\sqrt{2}} \sum_{\sigma=\pm} a^+_{\mathbf{x},\sigma} a^-_{\mathbf{x},\sigma}$ is the *charge density* and $\rho^1_{\mathbf{x}} = \frac{1}{\sqrt{2}} \sum_{\sigma=\pm} \sigma a^+_{\mathbf{x},\sigma} a^-_{\mathbf{x},\sigma}$ is the *spin density*. We also define the static correlation functions as

$$S^{\varepsilon}_{L,\beta}(x) = N^{\varepsilon}_{L,\beta}(\mathbf{x})|_{x_0=0^+}. \tag{5}$$

If there is no interaction $U = 0$, the two-point Schwinger function $g(\mathbf{x}-\mathbf{y})$ is given by

$$g(\mathbf{x}-\mathbf{y}) = \frac{1}{\beta L} \sum_{\mathbf{k}\in\mathcal{D}} \frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-\mathrm{i}k_0 + \mu - 1 + \cos k}, \tag{6}$$

with $\mathbf{k} = (k_0, k)$, $\mathcal{D} \equiv \mathcal{D}_L \times \mathcal{D}_\beta$, with $\mathcal{D}_L \equiv \{k = 2\pi n/L, n \in Z, -[L/2] \leqslant n \leqslant [(L-1)/2]\}$ and $\mathcal{D}_\beta \equiv \{k_0 = 2(n+1/2)\pi/\beta, n \in Z\}$. The density correlations for $U = 0$ are given by

$$N^{\varepsilon}_0(\mathbf{x}) = g(\mathbf{x})g(-\mathbf{x}), \tag{7}$$

and the static density correlations for $x \neq 0$ can be written as

$$S^{\varepsilon}_0(x) = \frac{1}{2\pi^2 x^2}\left(1 + \cos 2p^0_F x\right)\left[1 + O\left(\frac{1}{|x|}\right)\right], \tag{8}$$

where $\mu = 1 - \cos p^0_F$. Note that the dominant part of $S^{\varepsilon}_0(x)$ has an oscillating and a non-oscillating part, both decaying as $O(x^{-2})$ for large x.

Denoting by $\hat{S}^{\varepsilon}_0(k)$ the Fourier transform of $S^{\varepsilon}_0(x)$, we get, if $\varepsilon(k) = 1 - \cos k - \mu$,

$$\hat{S}^{\varepsilon}_0(k) = \int_{-\pi}^{\pi} \mathrm{d}p\, \chi(\varepsilon(p) < 0)\chi(\varepsilon(p+k) > 0), \tag{9}$$

that is, if $p^0_F \leqslant \frac{\pi}{2}$ for definiteness

$$\hat{S}^{\varepsilon}_0(k) = |k| \qquad |k| \leqslant 2p^0_F \qquad\qquad 2p^0_F \qquad \pi \geqslant |k| \geqslant 2p^0_F. \tag{10}$$

The first derivative is then discontinuous, that is $\partial_k \hat{S}^{\varepsilon}_0(k) = 1$ for $0 \leqslant k \leqslant 2p^0_F$, $\partial_k \hat{S}^{\varepsilon}_0(k) = -1$ for $-2p^0_F \leqslant k \leqslant 0$ and 0 otherwise.

As the spin and charge density correlations are directly linked to experiments, one is interested in what happens to such quantities when the interaction is switched on, especially close to 0, $\pm 2p_F$ where singularities are present. Such problem has been deeply investigated in the literature but no definite conclusions have been reached.

The classical perturbative renormalization group (RG) analysis in [13] shows that the repulsive Hubbard model has an attractive fixed point, up to third order in the perturbative expansion, in correspondence with the Mattis model [9], a solvable generalization to spinning

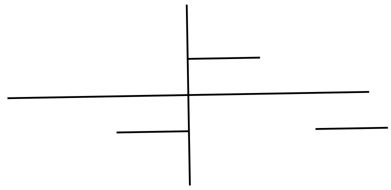

Figure 1. $\partial_k \hat{S}_0^\varepsilon(k)$ as a function of k; there are jumps at $k = 0, \pm 2p_F^0$, and the function is 2π periodic.

fermions of the Luttinger model solved in [10] and describing chiral fermions with linear dispersion relations. The Mattis model does not contain the most general interaction between spinning fermions, as the backward interactions are not included; they are instead present in the Hubbard model.

From the exact solution one can see, [1], that in the Mattis model the behaviour of the spin or charge density correlations for momenta close to $2p_F$ or 0 is quite different: in the first case the behaviour is anomalous with the appearance of a nontrivial critical index, while in the second case such behaviour is *unaffected* by the interaction.

Regarding the behaviour of the density correlations of the 1d Hubbard model, it is reasonable to guess that the behaviour close to $2p_F$ of the density correlations should be qualitatively the same as in the Mattis model: the nonvanishing critical index should not be cancelled by the presence of (dimensionally or marginal) irrelevant terms in the RG sense.

Much more delicate is, in contrast, the situation at zero momentum. The fact that the corresponding critical index is vanishing in the Mattis model is related, see [4, 11, 12], to the validity of certain ward identities based on symmetries under separate chiral and spin phase transformations which are however *not valid* in the Hubbard model, for the presence of backward interactions. It is true that iterating the RG the backscattering interactions are vanishing; however their convergence to zero is quite slow (non-summable) and this could produce a logarithmic divergence in the flow of the density renormalization, as is found in [7], unless suitable cancellations *at all orders* are found. On the other hand, the presence of logarithmic divergences would be in striking contrast with the metallic Luttinger liquid behaviour expected in the 1d repulsive Hubbard model.

New techniques based on a combination of exact RG methods with ward identities (modified by the presence of cut-offs) have been developed in [2, 3] for the analysis of interacting spinless fermions and extended to the Hubbard model in [8]; we apply such methods to study the spin and charge correlations of the 1d Hubbard model, proving the following result.

Theorem. *If $p_F^0 \neq \frac{\pi}{2}$, $\hat{v}(0) > \hat{v}(2p_F)$ and $\hat{v}(2p_F^0)U > 0$ and small enough, if $\varepsilon = 0, 1$ the static spin and charge density correlations can be written as*

$$S^\varepsilon(x) = \cos(2p_F x)\frac{1 + A_{1,\varepsilon}(x)}{2\pi^2 x^{2-\eta_\varepsilon}} + \frac{1 + A_{2,\varepsilon}(x)}{2\pi^2 x^2} + O\left(\frac{1}{|x|^{2+\vartheta}}\right), \tag{11}$$

where $|A_{i,\varepsilon}(x)| \leqslant CU$, $C_1 U \leqslant \eta_\varepsilon \leqslant C_2 U$, C, C_1, C_2, ϑ are positive constants.

The correlations $\hat{S}^\varepsilon(k)$ are bounded for all $k \in [-\pi, \pi]$, while their first derivatives $\partial_k \hat{S}^\varepsilon(k)$ are bounded for all $k \neq \pm 2p_F$. At $k = \pm 2p_F$, $\partial_k \hat{S}^\varepsilon(k)$ diverges as $|k - (\pm 2p_F)|^{-\eta_\varepsilon}$ and close to $k = 0$, we can write

$$\hat{S}^\varepsilon(k) = \hat{S}_0^\varepsilon(k) + U h^\varepsilon(k) \tag{12}$$

with $|h^\varepsilon(k)|, |\partial_k h^\varepsilon(k)| \leqslant C$.

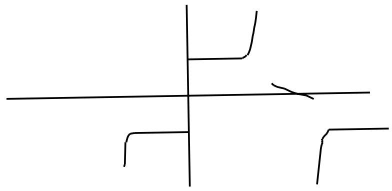

Figure 2. $\partial_k S^\varepsilon(k)$ as a function of k; there is a jump at $k = 0$ and a divergence at $\pm 2p_F$.

By comparing (8) with (11) we see that the oscillating part of the interacting density correlations has a different power law decay with respect to the free case, in contrast to what happens to the non-oscillating part.

Moreover, at $k = 0$, $\hat{S}^\varepsilon(k)$ has a cusp and $\partial_k \hat{S}^\varepsilon(k)$ has a finite discontinuity; the only effect of the interaction is to change the opening angle of the cusp and the width of the discontinuity, and no logarithmic divergences at zero momentum appear. In contrast, the interaction changes radically the singularity of $\hat{S}^\varepsilon(k)$ at $2p_F$: the first derivative has a power law divergence. For all other values of k, the static correlations and its first derivatives are continuous; possible singularities $\hat{S}^\varepsilon(k)$ or $\partial \hat{S}^\varepsilon(k)$ at $k \neq (0, \pm 2p_F)$ can possibly appear only at strong coupling.

The above theorem is proved in sections 2 and 3; in particular in section 2 we perform a multiscale analysis of the density correlations, and in section 3 we prove the cancellations related to the behaviour at zero momentum; section 4 is devoted to the conclusions.

2. Exact renormalization group approach

2.1. Grassmann integrals

We introduce two finite sets of anticommuting Grassmanian variables $\{\hat{\psi}^\pm_{\mathbf{k},\sigma}\}$ and $\{\mathrm{d}\hat{\psi}^\pm_{\mathbf{k},\sigma}\}$, $\mathbf{k} \in \mathcal{D}$ and $\sigma = \pm$, and we define an operation (*Grassmann integration*) in the following way:

$$\int \hat{\psi}^\pm_{\mathbf{k},\sigma}\, \mathrm{d}\hat{\psi}^\pm_{\mathbf{k},\sigma} = 1 \qquad \int \mathrm{d}\hat{\psi}^\pm_{\mathbf{k},\sigma} = 0. \tag{13}$$

We also define the *Grassmanian field* $\psi^\pm_{\mathbf{x},\sigma}$ as $\psi^\pm_{\mathbf{x},\sigma} = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}} \hat{\psi}^\pm_{\mathbf{k},\sigma}\, \mathrm{e}^{\pm\mathrm{i}\mathbf{k}\mathbf{x}}$. The density correlations of the Hubbard model can be obtained from a *generating function* $\mathcal{W}_\varepsilon$, $\varepsilon = 0, 1$ depending if the charge or spin correlations are considered, expressed by the following Grassmann integral:

$$\mathrm{e}^{\mathcal{W}_\varepsilon(\phi)} = \int P(\mathrm{d}\psi)\, \mathrm{e}^{-\mathcal{V}(\psi)+\int \mathrm{d}\mathbf{x}\phi_{\mathbf{x},\varepsilon}\frac{1}{\sqrt{2}}\sum_{\sigma=\pm}(\sigma)^\varepsilon \psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}}, \tag{14}$$

where $P(\mathrm{d}\psi)$ is the *fermionic integration*

$$P(\mathrm{d}\psi) = \left[\prod_{\sigma=\pm}\prod_{\mathbf{k}\in\mathcal{D}} \mathrm{d}\hat{\psi}^+_{\mathbf{k},\sigma}\, \mathrm{d}\hat{\psi}^-_{\mathbf{k},\sigma}\right] \exp\left[-\frac{1}{\beta L}\sum_\sigma\sum_{\mathbf{k}\in\mathcal{D}} \hat{\psi}^+_{\mathbf{k},\sigma}(-\mathrm{i}k_0 + \cos p_F - \cos k)\hat{\psi}^-_{\mathbf{k},\sigma}\right], \tag{15}$$

and the interaction $\mathcal{V}$ is

$$\mathcal{V}(\psi) = \nu \sum_\sigma \int \mathrm{d}\mathbf{x} \psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma} + \delta \int \mathrm{d}\mathbf{x} t_{x,y} \psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma} + U\sum_{\sigma,\sigma'}\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} v(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{x},\sigma'}\psi^-_{\mathbf{x},\sigma'}, \tag{16}$$

where $\int d\mathbf{x} \equiv \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x\in\Lambda}$, $U > 0$ (we are choosing $\hat{v}(2p_F) > 0$ for definiteness), $t_{x,y} = \frac{1}{2}\delta_{y,x+1} + \frac{1}{2}\delta_{x,y+1} - \cos p_F$ and $v(\mathbf{x}-\mathbf{y}) = \delta(x_0 - y_0)v(x-y)$. ν, δ are *counterterms* introduced in order to take into account that the interaction changes the value of the Fermi momentum and of the Fermi velocity with respect to the non-interacting case. Finally, $\phi_{\mathbf{x},\varepsilon}$ are *external fields*.

The density correlations are given by

$$N^{\varepsilon}(\mathbf{x}-\mathbf{y}) = \frac{\partial^2 \mathcal{W}_{\varepsilon}}{\partial \phi_{\mathbf{x},\varepsilon}\partial\phi_{\mathbf{y},\varepsilon}}|_{\phi=0}. \tag{17}$$

Our *exact* RG computation of the Hubbard model correlation functions is based on some elementary properties of Grassmann integrals, which we briefly recall.

(1) *Addition property.* If $P(d\psi^{(1)})$ and $P(d\psi^{(2)})$ are fermionic integrations with propagator g_1 and g_2, for any analytic function F it holds

$$\int P(d\psi)F(\psi) = \int P(d\psi^{(1)}) \int P(d\psi^{(2)})F(\psi^{(1)} + \psi^{(2)}), \tag{18}$$

where $P(d\psi)$ has propagator $g = g_1 + g_2$.

(2) *Invariance of exponentials.* If ϕ is a Grassmann variable

$$\int P(d\psi)\,e^{V(\psi+\phi)} = e^{\sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{E}^T(V;n)} \equiv e^{V'(\phi)}, \tag{19}$$

where $\mathcal{E}^T(V;n)$ is the truncated expectation with respect to $P(d\psi)$, that is the sum over all the connected Feynmann graphs obtained from n interactions V.

(3) *Change of integration.* If $P_g(d\psi)$ denotes the fermionic integration with covariance g,

$$\int P_g(d\psi)\,e^{\int d\mathbf{k}\nu(\mathbf{k})\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}} F(\psi) = \mathcal{N}\int P_{(g^{-1}+\nu)^{-1}}(d\psi)F(\psi). \tag{20}$$

2.2. Multiscale analysis

Let T^1 be the one-dimensional torus, $\|k-k'\|_{T^1}$ the usual distance between k and k' in T^1. We introduce a *scaling parameter* $\gamma > 1$ and a positive smooth function $\chi(\mathbf{k}')$, $\mathbf{k}' = (k', k_0)$, such that $\chi(\mathbf{k}') = 1$ if $|\mathbf{k}'| < t_0 \equiv a_0 v_0/\gamma$ and $\chi(\mathbf{k}') = 0$ if $|\mathbf{k}'| > a_0 v_0$, where $|\mathbf{k}'| = \sqrt{k_0^2 + (v_0\|k'\|_{T^1})^2}$ and $a_0 = \min\{p_F/2, (\pi - p_F)/2\}$, $v_0 = \sin p_F$. The above choice is such that the supports of $\chi(k-p_F, k_0)$ and $\chi(k+p_F, k_0)$ are disjoint and the C^∞ function on $T^1 \times R$:

$$f_{\text{u.v.}}(\mathbf{k}) \equiv 1 - \chi(k-p_F, k_0) - \chi(k+p_F, k_0) \tag{21}$$

is equal to 0, if $[v_0\|(|k| - p_F)\|_{T^1}]^2 + k_0^2 < t_0^2$. We define

$$g(\mathbf{x}-\mathbf{y}) = g^{(\text{u.v.})}(\mathbf{x}-\mathbf{y}) + g^{(\text{i.r.})}(\mathbf{x}-\mathbf{y}) \tag{22}$$

with

$$\begin{aligned} g^{(\text{u.v.})}(\mathbf{x}-\mathbf{y}) &= \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{f_{\text{u.v.}}(\mathbf{k})}{-ik_0 - \cos k + \cos p_F} \\ g^{(\text{i.r.})}(\mathbf{x}-\mathbf{y}) &= \frac{1}{L\beta}\sum_{\omega=\pm}\sum_{\mathbf{k}\in\mathcal{D}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\chi(k-\omega p_F, k_0)}{-ik_0 - \cos k + \cos p_F} \equiv \sum_{\omega=\pm} g_\omega^{(\leqslant 0)}(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{23}$$

where $g^{(\text{u.v.})}(\mathbf{x}-\mathbf{y})$ is the *ultraviolet* part of the propagator while $g^{(\text{i.r.})}(\mathbf{x}-\mathbf{y})$ is the infrared part; $\hat{f}_{\text{u.v.}}(\mathbf{k})$ has support far from the points $(0, \pm p_F)$ in which the free propagator is singular while $1 - \hat{f}_{\text{u.v.}}(\mathbf{k})$ has support around the two points $(0, \pm p_F)$.

We introduce two Grassmann variables $\psi^{(\mathrm{i.r.})}, \psi^{(\mathrm{u.v.})}$ with propagators $g^{(\mathrm{i.r.})}(\mathbf{x}-\mathbf{y})$, $g^{(\mathrm{u.v.})}(\mathbf{x}-\mathbf{y})$ and, by the addition property (18), we can write (14) as

$$\mathrm{e}^{\mathcal{W}_\varepsilon(\phi)} = \int P(\mathrm{d}\psi^{(i.r)})P(\mathrm{d}\psi^{(u.v)}) \times \mathrm{e}^{-\mathcal{V}(\psi^{(i.r)}+\psi^{(u.v)})+\frac{1}{\sqrt{2}}\int \mathrm{d}\mathbf{x}\phi_{\mathbf{x},\varepsilon}\sum_{\sigma=\pm}(\sigma)^\varepsilon(\psi^{(\mathrm{i.r.})+}_{\mathbf{x},\sigma}+\psi^{(\mathrm{u.v.})+}_{\mathbf{x},\sigma})(\psi^{(\mathrm{i.r.})-}_{\mathbf{x},\sigma}+\psi^{(\mathrm{u.v.})-}_{\mathbf{x},\sigma})}, \tag{24}$$

and we can integrate $\psi^{(\mathrm{u.v.})}$ obtaining, by (19),

$$\mathrm{e}^{\mathcal{W}_\varepsilon(\phi)} = \mathrm{e}^{S^{(1)}(\phi)}\int P(\mathrm{d}\psi^{(\mathrm{i.r.})})\,\mathrm{e}^{-\mathcal{V}^{(0)}(\psi^{(\mathrm{i.r.})})+\mathcal{B}^{(0)}(\psi^{(\mathrm{i.r.})},\phi)} \tag{25}$$

where $S^{(1)}(\phi), \mathcal{V}^{(0)}(\psi^{(\mathrm{i.r.})})$ and $\mathcal{B}^{(0)}(\psi^{(\mathrm{i.r.})},\phi)$ are sums over monomials in $\phi, \psi^{(\mathrm{i.r.})}$ and $\psi^{(\mathrm{i.r.})}, \phi$ respectively multiplied by kernels bounded and fast decaying; such regularity properties of the kernels are due to the fact that $g^{(\mathrm{u.v.})}(\mathbf{x}-\mathbf{y})$ is fast decaying for large distances, as a consequence that $\hat{f}_{\mathrm{u.v.}}(\mathbf{k})$ has support far away from the points $(0, \pm p_F)$ in which the denominator of $\hat{g}^{(\mathrm{u.v.})}(\mathbf{k})$ vanishes.

We write now, setting $k = k' + \omega p_F$, $\mathbf{k}' = (k', k_0)$:

$$g^{(\mathrm{i.r.})}_\omega(\mathbf{x}-\mathbf{y}) = \mathrm{e}^{\mathrm{i}\omega p_F(\mathbf{x}-\mathbf{y})} g^{(\leqslant 0)}_\omega(\mathbf{x}-\mathbf{y}), \tag{26}$$

where

$$g^{(\leqslant 0)}_\omega(\mathbf{x}-\mathbf{y}) = \sum_{h=-\infty}^{0} g^{(h)}_\omega(\mathbf{x}-\mathbf{y}) \tag{27}$$

and

$$g^{(h)}_\omega(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}} \mathrm{e}^{-\mathrm{i}\mathbf{k}'(\mathbf{x}-\mathbf{y})}\frac{f_h(\mathbf{k}')}{-\mathrm{i}k_0 - \cos(k'+\omega p_F)+\cos p_F} \tag{28}$$

and $f_h(\mathbf{k}') = \chi(\gamma^{-h}\mathbf{k}') - \chi(\gamma^{-h+1}\mathbf{k}')$ and $\chi(\mathbf{k}') = \sum_{h=-\infty}^{0} f_h(\mathbf{k}')$; finally we define $C_h^{-1}(\mathbf{k}') = \sum_{k=-\infty}^{h} f_k(\mathbf{k}')$. Note that in the support of $f_h(\mathbf{k}')$ the denominator of $\hat{g}^{(h)}_\omega(\mathbf{k}')$ is $O(\gamma^h)$.

We can introduce two Grassmann variables $\psi_+^{(\leqslant 0)}, \psi_-^{(\leqslant 0)}$ with Grassmann integration $P\big(\mathrm{d}\psi_+^{(\leqslant 0)}\big)$, $P(\mathrm{d}\psi_-^{(\leqslant 0)})$ and propagators $g_+^{(\leqslant 0)}(\mathbf{x}-\mathbf{y})$, $g_-^{(\leqslant 0)}(\mathbf{x}-\mathbf{y})$; by using again the addition property (18), we can rewrite (29) as

$$\mathrm{e}^{\mathcal{W}_\varepsilon(\phi)} = \mathrm{e}^{S^1(\phi)}\int P\big(\mathrm{d}\psi_+^{(\leqslant 0)}\big)P\big(\mathrm{d}\psi_-^{(\leqslant 0)}\big)\,\mathrm{e}^{-\mathcal{V}^{(0)}(\psi_+^{(\leqslant 0)},\psi_-^{(\leqslant 0)})+\mathcal{B}^{(0)}(\psi_+^{(\leqslant 0)},\psi_-^{(\leqslant 0)},\phi)}, \tag{29}$$

where $\mathcal{V}^{(0)}(\psi_+^{(\leqslant 0)}, \psi_-^{(\leqslant 0)})$ and $\mathcal{B}^{(0)}\big(\psi_+^{(\leqslant 0)}, \psi_-^{(\leqslant 0)}, \phi\big)$ are obtained from $\mathcal{V}^{(0)}(\psi^{(\mathrm{i.r.})})$ and $\mathcal{B}^{(0)}(\psi^{(\mathrm{i.r.})},\phi)$ by the replacement

$$\psi^{\pm(\mathrm{i.r.})}_{\mathbf{x},\sigma} \to \sum_{\omega=\pm}\mathrm{e}^{\pm\,\mathrm{i}\omega p_F(\mathbf{x}-\mathbf{y})}\psi^{(\leqslant 0)}_{\mathbf{x},\omega\sigma}. \tag{30}$$

Remark. After the integration of $\psi^{(\mathrm{u.v.})}$ one finds, see (29), that the system can be expressed in terms of chiral fields $\psi^{\pm}_{\omega,\sigma}$, where $\omega = \pm$, with approximately linear dispersion relation and an ultraviolet cut-off; this exact representation, based on the properties of Grassmann variables, substantiates the standard approximation of the Hubbard model with the *g-ology* model, see [13].

The analysis of (29) is done by a multiscale analysis based on the decomposition (27). Physically, $g^{(h)}_\omega(\mathbf{x}-\mathbf{y})$ represent the propagator 'at scale γ^h' and for any integer N and any $h \leqslant 0$:

$$\big|g^{(h)}_\omega(\mathbf{x}-\mathbf{y})\big| \leqslant \gamma^h\frac{C_N}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|^N}. \tag{31}$$

From the above bound we see that the scaling dimension of the fermionic fields is 1/2 and of the external field J is 1; hence $\int d\mathbf{x}\psi^+\psi^-$ has scaling dimension -1 (*relevant terms*), $\int d\mathbf{x}\psi^+\psi^-\psi^+\psi^-$ or $\int d\mathbf{x}\phi\psi^+\psi^-$ have dimension 0 (*marginal terms*) and all other terms have positive scaling dimension (*irrelevant terms*).

It is convenient to decompose $g_\omega^{(h)}(\mathbf{x}-\mathbf{y})$ as

$$g_\omega^{(h)}(\mathbf{x}-\mathbf{y}) = g_{\omega,L}^{(h)}(\mathbf{x}-\mathbf{y}) + r_\omega^{(h)}(\mathbf{x}-\mathbf{y}), \tag{32}$$

with

$$g_{\omega,L}^{(h)}(\mathbf{x}-\mathbf{y}) = \frac{1}{\beta L}\sum_{\mathbf{k}'\in\mathcal{D}} e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}\frac{f_h(\mathbf{k}')}{-ik_0+\omega k'} \tag{33}$$

and

$$\left|r_\omega^{(h)}(\mathbf{x}-\mathbf{y})\right| \leqslant \gamma^{2h}\frac{C_N}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|^N}, \tag{34}$$

that is $r_\omega^{(h)}(\mathbf{x}-\mathbf{y})$ has an extra small factor γ^h in the bound; note that $g_{\omega,L}^{(h)}(\mathbf{x}-\mathbf{y})$ is the propagator of Luttinger fermions with linear dispersion relation and bandwidth cut-off.

The multiscale integration of (29) can be done in an iterative way; assume that we have integrated the scales $0, -1, \ldots, h+1$ and that we have found, up to a constant

$$e^{S^{(h+1)}(\phi)}\int P_{Z_h,C_h}(d\psi^{\leqslant h})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},\phi)}, \tag{35}$$

where $Z_0 = 1$, $P_{Z_0,C_0}(d\psi^{\leqslant 0}) = P\big(d\psi_+^{(\leqslant 0)}\big)P(d\psi_-^{(\leqslant 0)})$, $\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) = \mathcal{V}^{(0)}\big(\psi_+^{(\leqslant 0)}, \psi_-^{(\leqslant 0)}\big)$, $\mathcal{B}^{(0)}(\psi^{(\leqslant 0)},\phi) = \mathcal{B}^{(0)}\big(\psi_+^{(\leqslant 0)}, \psi_-^{(\leqslant 0)},\phi\big)$ and, if $c_0 = \cos p_F$

$$\begin{aligned}
P_{Z_h,C_h}(d\psi^{(\leqslant h)}) = {} & \left[\prod_{\mathbf{k}'\in\mathcal{D}}\prod_{\omega,\sigma=\pm} d\hat\psi_{\mathbf{k}',\omega,\sigma}^{+(\leqslant h)}\, d\hat\psi_{\mathbf{k}',\omega,\sigma}^{-(\leqslant h)}\right] \\
& \times \exp\left\{-\frac{1}{\beta L}\sum_{\omega,\sigma=\pm}\sum_{\mathbf{k}'\in\mathcal{D}} Z_h C_h(\mathbf{k}')\hat\psi_{\mathbf{k}',\omega,\sigma}^{+(\leqslant h)}(-ik_0+\omega v_0\sin k'\right. \\
& \left. +c_0(\cos k'-1))\hat\psi_{\mathbf{k}',\omega,\sigma}^{-(\leqslant h)}\right\},
\end{aligned} \tag{36}$$

with

$$\begin{aligned}
\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) &= \sum_{n=1}^\infty\sum_{\underline{\omega},\underline{\sigma}}\int d\underline{\mathbf{x}}\prod_{i=1}^{2n}\psi_{\mathbf{x}_i,\omega_i,\sigma_i}^{(\leqslant h)\varepsilon_i} W_{2n,\underline{\sigma},\underline{\omega}}^{(h)}(\underline{\mathbf{x}}); \\
S^{(h+1)}(\phi) &= \sum_{m=1}^\infty\int d\underline{\mathbf{x}} S_m^{(h+1)}(\underline{\mathbf{x}})\prod_{i=1}^m\phi(\mathbf{x}_i) \\
\mathcal{B}^{(h)}(\psi^{(\leqslant h)},\phi) &= \sum_{m=1}^\infty\sum_{n=1}^\infty\sum_{\underline{\sigma},\underline{\omega}}\int d\underline{\mathbf{x}}\, d\underline{\mathbf{y}} B_{m,2n;\underline{\omega},\underline{\sigma}}^{(h)}(\underline{\mathbf{x}},\underline{\mathbf{y}})\prod_{i=1}^m\phi_{\mathbf{x}_i}(\mathbf{x}_i)\prod_{i=1}^{2n}\psi_{\mathbf{y}_i,\omega_i,\sigma_i}^{(\leqslant h)\varepsilon_i}.
\end{aligned} \tag{37}$$

We split the effective potential $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)}+\mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R} = 1-\mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator defined by its action on the kernels $\hat W_{2n,\underline{\omega}}^{(h)}$ (we denote by $\hat W_{2n,\underline{\omega}}^{(h)}$ the Fourier transform of $W_{2n,\underline{\omega}}^{(h)}$) in the following way:

(1) if $2n = 4$, we define

$$\mathcal{L}\hat W_{4,\underline{\sigma},\underline{\omega}}^{(h)}(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3') = \delta_{\sum_i\varepsilon_i\omega_i}\hat W_{4,\underline{\sigma},\underline{\omega}}^{(h)}(\mathbf{0},\mathbf{0},\mathbf{0}); \tag{38}$$

(2) if $2n = 2$, (in this case there is a non-zero contribution only if $\omega_1 = \omega_2$)

$$\mathcal{L}\hat{W}^{(h)}_{2,\sigma,\omega}(\mathbf{k}') = \hat{W}^{(h)}_{2,\sigma,\omega}(\mathbf{0}) + k_0\partial_{k_0}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{0}) + (\omega v_0 \sin k' + c_0(\cos k' - 1))\partial_{k'}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{0}), \quad \text{and} \tag{39}$$

(3) in all the other cases,

$$\mathcal{L}\hat{W}^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, \ldots, \mathbf{k}'_{2n-1}) = 0. \tag{40}$$

Note that the $\mathcal{L}$ operation acts on the terms with positive or scaling dimension; as

$$\mathcal{R}\hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}'_3) = \hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}'_3) - \hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{41}$$

it is easy to check that $\mathcal{R}$ decreases the size of $\mathcal{R}\hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}$ by a factor $\gamma^{h-h'} < 1$, where h' is the highest scale among the propagators contracted in $\hat{W}^{(h)}_{2n,\underline{\sigma},\underline{\omega}}$ and h is the scale of the external fields: such improvement in the scaling dimension of the relevant or marginal terms makes convergent the sums over scales. Note also that $\mathcal{L} = 0$ on the monomial multiplying $\psi^+_{\omega,\sigma}\psi^-_{-\omega,\sigma}\psi^+_{\omega,\sigma'}\psi^-_{-\omega,\sigma'}$; indeed they behave as irrelevant terms (despite dimensionally marginal) when $p_F \neq \frac{\pi}{2}$.

In the same way, we split $\mathcal{B}^{(h)}(\psi^{(\leqslant h)}, \phi)$ as $\mathcal{L}\mathcal{B}^{(h)} + \mathcal{R}\mathcal{B}^{(h)}$, where $\mathcal{L}$ is defined as its action on the kernels of $\mathcal{B}^{(h)}$ in the following way: $\mathcal{L}\hat{B}^{(h)}_{m,2n} = 0$, except when $m = 1, n = 1$, and

$$\mathcal{L}\hat{B}^{(h)}_{1,2}(\mathbf{p}, \mathbf{k}') = \hat{B}^{(h)}_{1,2}(0, 0). \tag{42}$$

We include the quadratic part of $\mathcal{L}\mathcal{V}^{(h)}$ given by $z_h \int \mathrm{d}\mathbf{k} \sum_{\omega,\sigma}(-\mathrm{i}k_0 + \omega \sin k' + c_0(\cos k' - 1))\hat{\psi}^+_{\mathbf{k}',\omega,\sigma}\hat{\psi}^-_{\mathbf{k}',\omega,\sigma}$ in the free integration; calling

$$\mathcal{L}\bar{\mathcal{V}}^h = \mathcal{L}\mathcal{V}^h - z_h \int \mathrm{d}\mathbf{k} \sum_{\omega,\sigma} \psi^+_{\mathbf{k}',\omega,\sigma}(-\mathrm{i}k_0 + \omega \sin k' + c_0(\cos k' - 1))\psi^-_{\mathbf{k}',\omega,\sigma}, \tag{43}$$

we obtain

$$\mathrm{e}^{S^{(h+1)}(\phi)} \int P_{\tilde{Z}_{h-1},C_h}(\mathrm{d}\psi^{(\leqslant h)})\, \mathrm{e}^{-\mathcal{L}\bar{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(\leqslant h)})} \times \mathrm{e}^{-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}) + \mathcal{L}\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},\phi) + \mathcal{R}\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},\phi)}, \tag{44}$$

where

$$\tilde{Z}_{h-1}(\mathbf{k}) = Z_h\big(1 + z_h C_h^{-1}(\mathbf{k})\big). \tag{45}$$

After rescaling the fields the rhs of (44) can be rewritten (up to a constant) as

$$\mathrm{e}^{S^{(h+1)}(\phi)} \int P_{Z_{h-1},C_{h-1}}(\mathrm{d}\psi^{(\leqslant h-1)}) \int P_{Z_{h-1},\tilde{f}_h^{-1}}(\mathrm{d}\psi^{(h)})\, \mathrm{e}^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) + \hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)},\phi)}, \tag{46}$$

where

$$Z_{h-1} = Z_h(1 + z_h); \qquad \tilde{f}_h(\mathbf{k}') = f_h(\mathbf{k}')\left[1 + \frac{z_h f_{h+1}(\mathbf{k}')}{1 + z_h f_h(\mathbf{k}')}\right] \tag{47}$$

and

$$\mathcal{L}\widehat{\mathcal{V}}^{(h)}(\psi) = \gamma^h \nu_h F_\nu + \delta_h F_\delta + \sum_{\sigma,\sigma'}\big[g_{1,h}F^{(h)}_{1,\sigma,\sigma'} + g_{2,h}F^{(h)}_{2,\sigma,\sigma'} + g_{4,h}F^{(h)}_{3,\sigma,\sigma'}\big], \tag{48}$$

where

$$F_\nu = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}}\sum_{\omega,\sigma}\hat{\psi}^+_{\mathbf{k}',\omega,\sigma}\hat{\psi}^-_{\mathbf{k}',\omega,\sigma}$$

$$F_\delta = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}}\sum_{\omega,\sigma}(\omega v_0 \sin k' + c_0(\cos k' - 1))\hat{\psi}^+_{\mathbf{k}',\omega,\sigma}\hat{\psi}^-_{\mathbf{k}',\omega,\sigma}$$

$$F_{1,\sigma,\sigma'} = \sum_{\omega} \int \mathrm{d}\mathbf{x} \psi^{+}_{\mathbf{x},\omega,\sigma} \psi^{-}_{\mathbf{x},-\omega,\sigma} \psi^{+}_{\mathbf{x},-\omega,\sigma'} \psi^{-}_{\mathbf{x},\omega,\sigma'} \tag{49}$$

$$F_{2,\sigma,\sigma'} = \sum_{\omega} \int \mathrm{d}\mathbf{x} \psi^{+}_{\mathbf{x},\omega,\sigma} \psi^{-}_{\mathbf{x},\omega,\sigma} \psi^{+}_{\mathbf{x},-\omega,\sigma'} \psi^{-}_{\mathbf{x},-\omega,\sigma'}$$

$$F_{4,\sigma,\sigma'} = \sum_{\omega} \int \mathrm{d}\mathbf{x} \psi^{+}_{\mathbf{x},\omega,\sigma} \psi^{-}_{\mathbf{x},\omega,\sigma} \psi^{+}_{\mathbf{x},\omega,\sigma'} \psi^{-}_{\mathbf{x},\omega,\sigma'},$$

and

$$g_{2,h} = \left[\frac{Z_h}{Z_{h-1}}\right]^2 \hat{W}^h_{4,\omega,\omega,-\omega,-\omega}(0,0,0), \qquad g_{1,h} = \left[\frac{Z_h}{Z_{h-1}}\right]^2 \hat{W}^h_{4,\omega,-\omega,\omega,-\omega}(0,0,0),$$

$$g_{4,h} = \left[\frac{Z_h}{Z_{h-1}}\right]^2 \hat{W}^h_{4,\omega,\omega,\omega,\omega}(0,0,0), \qquad \gamma^h \nu_h = \frac{Z_h}{Z_{h-1}} \hat{W}^h_2(0)$$

$$\delta_h = \frac{Z_h}{Z_{h-1}} \left[\partial_{k_0} \hat{W}^h_2(0) - \partial_k \hat{W}^h_2(0)\right].$$

By construction

$$\nu_0 = \nu + O(U), \qquad \delta_0 = \delta + O(U), \qquad g_{1,0} = U\hat{v}(2p_F) + O(U^2),$$
$$g_{2,0} = U\hat{v}(0) + O(U^2), \qquad g_{4,0} = U\hat{v}(0) + O(U^2). \tag{50}$$

In writing (49), we have used that the kernels in the effective potential with four external lines $\int \mathrm{d}\underline{\mathbf{x}} \psi^{+}_{\mathbf{x}_1,\omega_1,\sigma} \psi^{-}_{\mathbf{x}_2,\omega_2,\sigma} \psi^{+}_{\mathbf{x}_3,\omega_3,\sigma'} \psi^{-}_{\mathbf{x}_4,\omega_4,\sigma'} W^h_{\sigma,\sigma'}(\underline{\mathbf{x}})$ are such that $W^h_{\sigma,\sigma} = W^h_{\sigma,-\sigma}$ by the spin symmetry of the Hubbard model.

Moreover,

$$\mathcal{L}\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}, \phi) = \frac{1}{\sqrt{2}} Z^{(2,\varepsilon)}_{h-1} \int \mathrm{d}\mathbf{x} \phi_\varepsilon(\mathbf{x}) \sum_{\sigma,\omega} (\sigma)^\varepsilon \psi^{(\leqslant h)+}_{\mathbf{x},\omega,\sigma} \psi^{(\leqslant h)-}_{\mathbf{x},\omega,\sigma}$$
$$+ \frac{1}{\sqrt{2}} Z^{(1,\varepsilon)}_{h-1} \int \mathrm{d}\mathbf{x} \phi_\varepsilon(\mathbf{x}) \sum_{\sigma,\omega} (\sigma)^\varepsilon \, \mathrm{e}^{2\mathrm{i}\omega p_F} \psi^{(\leqslant h)+}_{\mathbf{x},\omega,\sigma} \psi^{(\leqslant h)-}_{\mathbf{x},-\omega,\sigma}, \tag{51}$$

where, if $i = 1, 2$, $Z^{(i)}_{h-1,\varepsilon} = \hat{B}^{h(i)}_{1,2;\varepsilon}(0,0)$ and

$$\frac{Z^{(i)}_{h-1,\varepsilon}}{Z^{(i)}_{h,\varepsilon}} = 1 + z^{(i)}_{h,\varepsilon}, \tag{52}$$

with $z^{(i)}_{h,\varepsilon} = O(\vec{v}_k)$. In (51), we have used that $\mathcal{V}$ is invariant under spin reflection while the source term in (9) acquires a sign $(\sigma)^\varepsilon$. We integrate then ψ^h and the description of the iterative procedure is then completed.

The above procedure generates an expansion for $S^{(h)}_{2n,\underline{\sigma},\underline{\omega}}$, $W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}$, $B^{(h)}_{m,2n;\underline{\omega},\underline{\sigma}}$ in terms of the *running coupling constants* $\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0$, with $\vec{v}_k = (\nu_k, \delta_k, g_{1,k}, g_{2,k}, g_{4,k})$.

It is possible to prove, (see theorem (3.12) of [2] to which we refer for a detailed proof) the following result.

Theorem. *There exists* ε_h *such that, for* $p_F \neq \frac{\pi}{2}$, $S^{(h)}_{2n,\underline{\sigma},\underline{\omega}}$, $W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}$, $B^{(h)}_{m,2n;\underline{\omega},\underline{\sigma}}$ *are analytic as functions of the running coupling constants* $\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0$ *for* $\sup_{k\geqslant h} |\vec{v}_k| \leqslant \varepsilon_h$ *and*

$$\int \mathrm{d}\underline{\mathbf{x}} \left| W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\underline{\mathbf{x}}) \right| \leqslant L\beta C^n \varepsilon^n_h \gamma^{-h(-2+n)}$$
$$\int \mathrm{d}\underline{\mathbf{x}}\, \mathrm{d}\underline{\mathbf{y}} \left| B^{(h)}_{m,2n;\underline{\omega},\underline{\sigma}} \right| (\underline{\mathbf{x}}, \underline{\mathbf{y}}) \leqslant L\beta C^{n+m} \varepsilon^{n+m}_h \gamma^{-h(-2+n+m)}. \tag{53}$$

The key technical ingredient to prove the above bounds is the following classical formula for the fermionic truncated expectations:

$$\mathcal{E}_h^T(\tilde{\psi}(P_1)\cdots\tilde{\psi}(P_n)) = \sum_T \left(\prod_{l\in T} g_l^{(h)}\right) \int \mathrm{d}P_T(t)\,\det G^T, \tag{54}$$

where T is a set of lines form a tree between the clusters of points $P_1, \ldots, P_n$, $\mathrm{d}P_T$ is a suitable normalized probability measure. If $\mathcal{E}_h^T$ is the truncated expectation with respect to ψ^h, by the *Gram–Hadamard* inequality for determinants it follows that $|\det G^T| \leqslant \gamma^{(\sum_i |P_i|-n)h}$. The determinant bound allows us to exploit the cancellations due to the anticommutativity; there are no $n!$, destroying the convergence, which one could find bounding each Feynmann graph.

From the above construction it is clear that $\vec{v}_h$ verify an iterative equation of the form

$$\vec{v}_{h-1} = \vec{\beta}_{\vec{v}}^h(\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0), \tag{55}$$

which is the analogue of the beta function in our exact RG analysis; note, however, that the lhs of (55) does not depend only on $\vec{v}_h$ but also on the running coupling constants at any scale $\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0$. The function $\vec{\beta}_{\vec{v}}^h$ is a power series in $\vec{v}_{h+1}, \ldots, \vec{v}_0$ which is convergent if $\sup_{k\geqslant h} |\vec{v}_k| \leqslant \varepsilon_h$. Note also that $\vec{\beta}_{\vec{v}}^h$ is given by a sum of terms obtained from truncated expectations at different scales from h to a certain scale $k \geqslant h$; with respect to bound (53), each of such terms is bounded by an extra factor $\gamma^{\frac{1}{2}(h-k)}$. This property is called *short memory* and explains that contributions to the running coupling constants at a scale h coming from the scale much greater than h are exponentially depressed, see [2] for details.

The boundedness of the flow of the running coupling constants $\vec{v}_h$ in the repulsive Hubbard model is a consequence of intricate cancellations at all orders in the beta function proved in [8], extending a previous result for spinless fermions found in [3]. While it is not difficult to check such cancellations at first orders, the cancellations must be proved *at all orders* in order to prove the boundedness of the flow, and this is done by implementing at each RG iteration Schwinger–Dyson equations and (modified) ward identities based on local-phase transformation and taking into account the effect of the cut-offs introduced in the RG analysis.

The following result has been proved in [8] (see theorem 4 of [8]).

Theorem. *For $U\hat{v}(2p_F) > 0$ and small enough it is possible to choose $\nu = O(U)$ and $\delta = O(U)$ such that, for any h, $|\nu_h| \leqslant CU\gamma^{\frac{h}{2}}$, $|\delta_h| \leqslant CU\gamma^{\frac{h}{2}}$ and for a positive constant a,*

$$\begin{gathered} |g_{2,h} + g_{1,h}/2 - g_{2,0} - g_{1,0}/2| \leqslant CU^2 \qquad |g_{4,h} - g_{4,0}| \leqslant CU^2 \\ 0 < g_{1,h} \leqslant \frac{g_{1,0}}{1 - a/3g_{1,0}h} \qquad |\vec{v}_{h-1} - \vec{v}_h| \leqslant C\left[\left[\frac{g_{1,0}}{1 - a/3g_{1,0}h}\right]^2 + U\gamma^{\frac{h}{2}}\right]. \end{gathered} \tag{56}$$

The above equation explains that $g_{1,h}$ is vanishing as $h \to -\infty$ while $g_{2,h}$, $g_{4,h}$ remains close to their initial value.

Remark. The above analysis says that iterating the RG one gets an effective theory of spinning chiral fermions with essentially linear (up to corrections vanishing iterating the RG) dispersion relation, and three quartic effective interactions, with couplings $g_{1,h}$, called the *backward* interaction, and $g_{2,h}$, $g_{4,h}$, called the *forward* interaction. Repeating a similar analysis for the Mattis model, one gets a similar effective theory with $\nu_h = \delta_h = g_{1,h} = 0$. Note that the forward scattering terms are invariant under separate phase transformations for each chirality and spin, that is $\psi^{\pm}_{\omega,\sigma} \to \mathrm{e}^{\pm\mathrm{i}\alpha_{\omega,\sigma}}\psi^{\pm}_{\omega,\sigma}$ with $\alpha_{\omega,\sigma}$ being an arbitrary function of ω and σ; in contrast the backward interaction is invariant only under phase transformations independent from the chirality; the solvability of the Mattis model is connected to the absence of backward scattering interactions.

2.3. The effective renormalizations

We have now to discuss the flow of the effective renormalizations Z_h, $Z_h^{(1)\varepsilon}$, $Z_h^{(2)\varepsilon}$. The flow equation for Z_h is

$$Z_{h-1} = Z_h + \sum_{k=h}^{0} \beta_z^{h,k}(\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0) Z_k \equiv Z_h\big(1 + \beta_z^h\big), \tag{57}$$

with $\left|\beta_z^{h,k}\right| \leqslant CU^2\gamma^{\frac{1}{2}(h-k)}$; from an explicit computation, if a is a constant and using that $|\nu_h| \leqslant CU\gamma^{\frac{h}{2}}$, $|\delta_h| \leqslant CU\gamma^{\frac{h}{2}}$,

$$\beta_z^h = a\big(g_{2,h}^2 + g_{4,h}^2 + g_{2,h}^2 - g_{1,h}g_{2,h}\big) + O(U^2|g_{1,h}|) \tag{58}$$

so that, by (57), (58),

$$\gamma^{-c_1U^2h} \leqslant Z_h \leqslant \mathrm{e}^{-c_2U^2h}. \tag{59}$$

The flow equation for $Z_{h-1}^{(1),\varepsilon}$ is given by

$$\frac{Z_{h-1}^{(i),\varepsilon}}{Z_{h-1}} = \frac{Z_h^{(i),\varepsilon}}{Z_h}\big[1 + \beta_i^h(\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0)\big], \tag{60}$$

with

$$1 + \beta_i^h = \frac{1 + z_h^{(i)}}{1 + z_h}, \tag{61}$$

and, if b is a constant

$$\beta_1^h = ag_{2,h} - ag_{1,h} + O(U^2) + O(U\gamma^{\frac{h}{2}}) \tag{62}$$

so that, using $g_{2,h} > g_{1,h}$ as $\hat{v}(0) > \hat{v}(2p_F)$:

$$\gamma^{-c_1Uh} \leqslant Z_h^{(1),\varepsilon} \leqslant \mathrm{e}^{-c_2Uh}. \tag{63}$$

The analysis of $Z_{h-1}^{(2),\varepsilon}$ is more delicate; the main point of the present paper is indeed to prove that

$$\frac{Z_h^{(2),\varepsilon}}{Z_h} = 1 + O(U), \tag{64}$$

which says that the density renormalization $Z_h^{(2),\varepsilon}$ is proportional to the wavefunction renormalization Z_h.

Remark. Equation (64) is in contrast with [6] in which it was found that $Z_h^{(2)\varepsilon}/Z_h$ is diverging as $h \to -\infty$ as $O(U^2|h|)$.

In order to prove (64) we can decompose β_2^h in (60) as sum of two terms; defining $\vec{g}_k = (g_{1,k}, g_{2,k}, g_{4,k})$ we have

$$\beta_2^h(\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0) = \beta_{2,a}^h(\vec{g}_h, \vec{g}_{h+1}, \ldots, \vec{g}_{-1}) + R_2^h(\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0), \tag{65}$$

where we include in $\beta_{1,a}^h$ only the terms contributing to β_1^h obtained contracting the quartic part of $\mathcal{LV}^{(k)}$, $k \leqslant -1$ with the dominant part of the propagator $g_L^{(k)}(\mathbf{x})$, $k \leqslant -1$, in $R_2^{(h)}$ are the remaining terms. One can check that such decomposition respects the determinant structure of the truncated expectations, see [2].

The following bound holds:

$$\big|R_2^h(\vec{v}_h, \vec{v}_{h+1}, \ldots, \vec{v}_0)\big| \leqslant CU\gamma^{\frac{h}{2}}. \tag{66}$$

The above bound follows from the fact that by the definition R_2^h is given by a sum of terms either obtained contracting the quadratic part of $\mathcal{L}\mathcal{V}^{(k)}$ at the same scale k, or such that the contraction has been done through the propagator $r_\omega^{(k)}(\mathbf{x})$; by using the short memory property, the fact that $|\nu_k|, |\delta_k| \leqslant CU\gamma^{\frac{k}{2}}$ and bound (34), (66) follows.

As R_2^h is *summable* with h, the proof of (64) is reduced to the proof of summability of β_2^h; we can write

$$\beta_{2,a}^h(\vec{g}_h, \vec{g}_{h+1}, \dots, \vec{g}_{-1}) = \beta_{2,a}^h(\vec{g}_h, \vec{g}_h, \dots, \vec{g}_h) + \sum_{k\geqslant h} D_{h,k}^2(\vec{g}_h, \vec{g}_{h+1}, \dots, \vec{g}_{-1}), \tag{67}$$

with

$$D_{h,k}^2(\vec{g}_h, \vec{g}_{h+1}, \dots, \vec{g}_{-1}) = \beta_{2,a}^h(\vec{g}_h, .., \vec{g}_h, \vec{g}_k \dots, \vec{g}_{-1}) - \beta_{2,a}^h(\vec{g}_h, .., \vec{g}_{k+1}, \vec{g}_k \dots, \vec{g}_{-1}). \tag{68}$$

By the short memory property and (56)

$$\begin{aligned}\sum_{k\geqslant h} \left|D_{h,k}^1\right| &\leqslant C_1 U \sum_{k\geqslant h} \gamma^{\frac{h-k}{2}} |\vec{g}_h - \vec{g}_k| \leqslant C_2 U \sum_{k\geqslant h} \gamma^{\frac{h-k}{2}} \sum_{i=k}^{h} \left[\frac{1}{k^2} + U\gamma^{\frac{k}{2}}\right] \\ &\leqslant C_3 U \sum_{k\geqslant h} \gamma^{\frac{h-k}{2}} |h-k| \frac{1}{k^2} \leqslant C_4 U \sum_{k\geqslant h} \gamma^{\frac{h-k}{4}} \frac{1}{k^2} \leqslant \frac{C_5 U}{h^2},\end{aligned} \tag{69}$$

so that the second addend of the rhs of (69) is summable.

The first addend of the rhs of (69) can be written as

$$\beta_{2,a}^h(\vec{g}_h, \vec{g}_h, \dots, \vec{g}_h) = \sum_{m_1,m_2,m_3} c_{m_1,m_2,m_3}^h (g_{1,h})^{m_1} (g_{2,h})^{m_2} (g_{4,h})^{m_2} \tag{70}$$

and, by the short memory property, $c_{m_1,m_2,m_3}^h = c_{m_1,m_2,m_3} + O(\gamma^{\frac{h}{2}})$. The coefficients c_{m_1,m_2,m_3} are obtained by the truncated expectations of m_1^p interaction $F_{1,\sigma,\sigma}$, m_1^o interaction $F_{1,\sigma,-\sigma}$, m_2^p interaction $F_{2,\sigma,\sigma}$, m_2^o interaction $F_{2,\sigma,-\sigma}$ and m_3 interaction $F_{4,\sigma,-\sigma}$ so that we can write

$$c_{m_1,m_2,m_3} = \sum_{m_1^o+m_1^p=m_1} \sum_{m_2^o+m_2^p=m_2} \sum_{m_3} c_{m_1^o,m_1^p,m_2^o,m_2^p,m_3}. \tag{71}$$

Note that

$$c_{0,m_2,m_3} = \sum_{m_2^o+m_2^p=m_2} \sum_{m_3} c_{0,0,m_2^o,m_2^p,m_3}, \qquad c_{1,m_2,m_3} = \sum_{m_2^o+m_2^p=m_2} \sum_{m_3} c_{0,1,m_2^o,m_2^p,m_3}. \tag{72}$$

The second part of (72) follows from the fact that there are no possible contributions obtained contracting $\psi_{\omega,\sigma}^+ \psi_{-\omega,\sigma}^- \psi_{-\omega,-\sigma}^+ \psi_{\omega,-\sigma}^-$ and any number of F_2, F_4, as the fields to be contracted would be, if the external lines have index (ω,σ), $n_1 + 1 - 2$ fields (ω,σ), $n_2 + 1$ fields $(\omega,-\sigma)$, $n_3 + 1$ fields $(-\omega,\sigma)$, $n_4 + 1$ fields $(-\omega,-\sigma)$, with n_1, n_2, n_3, n_4 *even*, as they are the number of fields coming from the interactions F_2^k and F_4^k which are bilinear in the densities of fermions of label (ω',σ').

Note finally that

$$F_{1,\sigma,\sigma}^h = -F_{2,\sigma,\sigma}^h, \tag{73}$$

and this implies

$$c_{0,1,m_2^o,m_2^p,m_3} = -c_{0,0,m_2^o,m_2^p+1,m_3}. \tag{74}$$

We will prove in section 2 that, for any m_2, m_3,

$$c_{0,m_2,m_3} = 0, \qquad c_{1,m_2,m_3} = 0. \tag{75}$$

Hence

$$e^{c_2 U \sum_{k=h}^{0}[\gamma^{\frac{k}{2}}+|k|^{-2}]} \leqslant \frac{\left|Z_{h-1}^{(2)\varepsilon}\right|}{|Z_{h-1}|} \leqslant e^{c_1 U \sum_{k=h}^{0}[\gamma^{\frac{k}{2}}+|k|^{-2}]} \tag{76}$$

from which (64) follows.

Remark. Even if the backward scattering interaction $g_{1,h} \rightarrow_{h\to-\infty} 0$, one also needs the second part of (75) to prove the finiteness of $Z_h^{(2)\varepsilon}/Z_h$; if $c_{0,m_2,m_3} = 0$ but $c_{1,\bar m_2,\bar m_3} = 0$ (but it is vanishing for all $m_2 + m_3 < \bar m_2 + \bar m_3$), one would obtain that $Z_h^{(2)\varepsilon}/Z_h$ is diverging as $h \to -\infty$ as $O(U^{\bar m_2+\bar m_3}|h|)$.

2.4. Density correlations

From the previous analysis, we have obtained a *convergent* expansion for the density correlations, which can be written as

$$N^\varepsilon(\mathbf{x}) = \cos(2p_F x)H_\varepsilon^a(\mathbf{x}) + H_\varepsilon^b(\mathbf{x}) + H_\varepsilon^c(\mathbf{x}), \tag{77}$$

where

$$H_\varepsilon^a(\mathbf{x}) = \sum_{h=-\infty}^{0}\left[\frac{Z_h^{(1),\varepsilon}}{Z_h}\right]^2\left[\sum_{\omega=\pm} g_\omega^{(h)}(\mathbf{x})g_{-\omega}^{(h)}(-\mathbf{x}) + \bar\Omega_a^{(h)}(\mathbf{x})\right] \tag{78}$$

$$H_\varepsilon^b(\mathbf{x}) = \sum_{h=-\infty}^{0}\left[\frac{Z_h^{(2),\varepsilon}}{Z_h}\right]^2\left[\sum_{\omega=\pm} g_\omega^{(h)}(\mathbf{x})g_{\omega}^{(h)}(-\mathbf{x}) + \bar\Omega_b^{(h)}(\mathbf{x})\right] \tag{79}$$

$$H_\varepsilon^c = \sum_{h=-\infty}^{0} \bar\Omega_c^{(h)}(\mathbf{x}), \tag{80}$$

and where we include in $\bar\Omega_{a,\omega}^{(h)}(\mathbf{x})$, $\bar\Omega_{b,\omega}^{(h)}(\mathbf{x})$ only the terms obtained contracting the quartic part of $\mathcal{L}\mathcal{V}^{(k)}$, $k \leqslant 0$ with propagators $g_L^{(h)}(\mathbf{r})$ and with two vertices $Z_k^{(1)}$ or $Z_k^{(2)}$ respectively; it holds that for $i = a, b$,

$$\left|\partial^n \bar\Omega_{a,b}^{(h)}(\mathbf{x})\right| \leqslant \gamma^{(2+n)h} U \frac{C_N}{1+(\gamma^h|\mathbf{x}|)^N} \tag{81}$$

as a consequence of the fact that all the oscillating factor $e^{\pm i p_F x_i}$ cancel out as $\sum_i \varepsilon_i\omega_i$ in the quartic monomials in $\mathcal{L}\mathcal{V}^{(k)}$. Moreover,

$$\left|\bar\Omega_c^{(h)}(\mathbf{x})\right| \leqslant \gamma^{\frac{5}{2}h} U \frac{C_N}{1+(\gamma^h|\mathbf{x}|)^N}, \tag{82}$$

and the extra $\gamma^{\frac{h}{2}}$ in the above bound is due to the short memory property, together with the fact that $\bar\Omega_c^{(h)}(\mathbf{x},\mathbf{y})$ is the sum of terms containing or a ν_k, δ_h (remember that $|\nu_k| \leqslant CU\gamma^{\frac{k}{2}}$, $|\delta_k| \leqslant CU\gamma^{\frac{k}{2}}$), or $r_\omega^{(k)}(\mathbf{x}-\mathbf{y})$ (whose bound (34) has a $\gamma^{\frac{k}{2}}$ more respect to $g_\omega^{(k)}(\mathbf{x}-\mathbf{y})$) or $g_\omega^{(\text{u.v.})}(\mathbf{x}-\mathbf{y})$. Assuming (64), we can obtain the properties of the Fourier transform of $N^\varepsilon(\mathbf{x})$. From (78), (81) and (82)

$$\left|H_\varepsilon^c(\mathbf{x})\right| \leqslant \frac{CU}{1+|\mathbf{x}|^{\frac{5}{2}}}, \qquad \left|\partial^n H_\varepsilon^a(\mathbf{x})\right| \leqslant \frac{C}{1+|\mathbf{x}|^{2-\eta_\varepsilon+n}}, \qquad \left|\partial^n H_\varepsilon^b(\mathbf{x})\right| \leqslant \frac{C}{1+|\mathbf{x}|^{2+n}}, \tag{83}$$

with $\eta_\varepsilon = O(U)$ and positive and $H_\varepsilon^a(\mathbf{x}) = H_\varepsilon^a(-\mathbf{x})$, $H_\varepsilon^b(\mathbf{x}) = H_\varepsilon^b(-\mathbf{x})$; this is due to the fact that $H_\varepsilon^a(\mathbf{x})$ and $H_\varepsilon^b(\mathbf{x})$ are sum over an even number of odd propagators

$g^{(h)}_{\omega,L}(\mathbf{x}) = -g\omega, L^{(h)}(-\mathbf{x})$. From (83), we see that $H^a_\varepsilon(\mathbf{x})$ and $H^b_\varepsilon(\mathbf{x})$ are free of oscillations, and the only oscillating factor in the first two addends of (77) is the prefactor $\cos 2p_F x$ in the first addend; on the other hand, $H^c_\varepsilon(\mathbf{x})$ has oscillating factors with period $\frac{2\pi}{2np_F}$ with any n but it has a much faster decay for $|\mathbf{x}| \to \infty$. The asymptotic formula (11) then follows.

We discuss now the properties of the one-dimensional Fourier transform of $N^\varepsilon(\mathbf{x})|_{x_0=0}$. Let us consider, for $i = a, b$, $H^i_\varepsilon(\mathbf{x}) \equiv H^i_\varepsilon(x, x_0)$; the Fourier transform of $H^i_\varepsilon(x, 0)$ is of course bounded by (83) and its derivative is given by

$$\begin{aligned}\left|\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{i}x H^i_\varepsilon(x,0)\right| &\leqslant \left|\frac{1}{k}\int \mathrm{d}x[\mathrm{e}^{\mathrm{i}kx}-1]\partial_x\big[xH^i_\varepsilon(x,0)\big]\right| \\ &\leqslant \left|\frac{1}{k}\int_{|x|\geqslant|k|^{-1}} \mathrm{d}x[\mathrm{e}^{\mathrm{i}kx}-1]\partial_x\big[xH^i_\varepsilon(x,0)\big]\right. \\ &\quad + \left|\frac{1}{k}\int_{|x|\leqslant|k|^{-1}} \mathrm{d}x[\mathrm{e}^{\mathrm{i}kx}-1-\mathrm{i}kx]\partial_x\big[xH^i_\varepsilon(x,0)\big]\right|, \end{aligned} \tag{84}$$

where we used the fact that $\partial_x\big[xH^i_\varepsilon(x,0)\big]$ is an even function of x. Hence, if $|k| \geqslant 1$, $\big|\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} x H^i_\varepsilon(x,0)\big| \leqslant C|k|^{-1}$, while, if $0 < |k| \leqslant 1$:

$$\left|\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} x H^a_\varepsilon(x,0)\right| \leqslant C[1+|k|^{-\eta_\varepsilon}] \tag{85}$$

and

$$\left|\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} x H^b_\varepsilon(x,0)\right| \leqslant C. \tag{86}$$

3. Ward identities with cut-off

3.1. The auxiliary model

In order to complete the proof of (64) we have still to prove (75) for any m_2, m_3.

Let us recall first how the proof of the analogue of (64) was achieved in the *spinless* Hubbard model, see [2, 3]. The analysis of the spinless Hubbard model is done through a multiscale analysis similar to the one of the previous sections, with the only difference that the quartic part of $\mathcal{LV}^{(h)}$ is given by $\lambda_h \int \mathrm{d}\mathbf{x} \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-}$, and that, for any $h, \lambda_h = \lambda_0 + O\big(\lambda_0^2\big)$; moreover the first addend of the rhs of (51) is replaced by $Z^{(2)}_{h-1}\int \mathrm{d}\mathbf{x}\phi(\mathbf{x})\sum_\omega \psi^{(\leqslant h)+}_{\mathbf{x},\omega}\psi^{(\leqslant h)-}_{\mathbf{x},\omega}$. Again $\frac{Z^{(2)}_h}{Z_h}$ verifies a flow equation similar to (60) with β^h_2 which can be decomposed as in (65), (67), with $\beta^h_{2,a}(\lambda_h, \lambda_h, \ldots, \lambda_h) = \sum_m c^h_m \lambda^m_h$. In [3], it was proved that $c^h_m = c_m + O(\gamma^{\frac{h}{2}})$ and $c_m = 0$, that is β^h_2 is asymptotically vanishing, and from this property the analogous of bound (64) trivially follows. The proof that $c_m = 0$ was obtained, see [3], through the introduction of an auxiliary model directly expressed in terms of chiral fermions $\psi^\pm_{\mathbf{x},\omega}$, $\omega = \pm$, with linear dispersion relation, ultraviolet and infrared cut-offs and interaction equal to $\lambda_0 \int \mathrm{d}\mathbf{x}\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-}$. The auxiliary model is chosen to be essentially equivalent in the infrared to the spinless Hubbard model; more exactly, the correlations of the two models can be expressed in terms of a set of quartic running coupling constants and effective renormalizations which have the *same* beta function up to corrections $O(\gamma^{\frac{h}{2}})$; in particular the coefficients c_m are the same in the two models. On the other hand, the advantage of the auxiliary model, with respect to the Hubbard model, is that it is *exactly* invariant under the global phase symmetry

$$\psi^\pm_{\mathbf{x},\omega} \to \mathrm{e}^{\pm\mathrm{i}\alpha_\omega}\psi^\pm_{\mathbf{x},\omega}, \tag{87}$$

and from such invariance a ward identity can be derived for the reference model which explains that $Z_h^{(2)\varepsilon}$ and Z_h are essentially proportional; this property implies that $c_m = 0$ for any m.

In the spinning case, we could try to follow the same strategy introducing an auxiliary model with quartic running coupling constants and effective renormalizations with beta functions asymptotically equal to those of the spinning Hubbard model; such a model would describe chiral fermions with linear dispersion relation, momentum cut-off, and local interaction given by the quartic part of $\mathcal{L}\mathcal{V}^{(0)}$ (48); the problem is however that $\mathcal{L}\mathcal{V}^{(0)}$ is *not invariant* under the generalization to the spinning case of (87), namely

$$\psi^{\pm}_{\mathbf{x},\omega,\sigma} \to \mathrm{e}^{\pm \mathrm{i}\alpha_{\omega,\sigma}} \psi^{\pm}_{\mathbf{x},\omega,\sigma}. \tag{88}$$

On the other hand, in the spinning case we do need to prove that $c_{m_1,m_2,m_3} = 0$, for any m_1, m_2, m_3 in order to get (64), but the weaker property (75), as $g_{1,h} = O(h^{-1})$. We will consider then the following *auxiliary model*, whose generating function is given by

$$\mathrm{e}^{\mathcal{H}_\varepsilon(\phi,J)} = \int P(\mathrm{d}\psi) \exp\left(-\mathcal{V}(\psi) + \frac{1}{\sqrt{2}} \sum_{\omega,\sigma} \int \mathrm{d}\mathbf{x} \phi_{\mathbf{x},\varepsilon}(\sigma)^\varepsilon \psi^+_{\omega,\sigma,\mathbf{x}} \psi^-_{\omega,\sigma,\mathbf{x}} \right.$$
$$\left. + \sum_{\omega,\sigma} \int \mathrm{d}\mathbf{x} [\psi^+_{\omega,\sigma,\mathbf{x}} J^-_{\omega,\sigma,\mathbf{x}} + J^+_{\omega,\sigma,\mathbf{x}} \psi^-_{\omega,\sigma,\mathbf{x}}]\right), \tag{89}$$

where

$$P(\mathrm{d}\psi) = \left[\prod_{\sigma,\omega=\pm} \prod_{\mathbf{k}} \mathrm{d}\hat{\psi}^+_{\mathbf{k},\omega,\sigma} \mathrm{d}\hat{\psi}^-_{\mathbf{k},\omega,\sigma}\right] \exp -\frac{1}{\beta L} \sum_{\omega,\sigma} \sum_{\mathbf{k}\in\mathcal{D}} \hat{\psi}^+_{\mathbf{k},\omega,\sigma} C_k(\mathbf{k})^{-1} (-\mathrm{i}k_0 + \omega k) \hat{\psi}^-_{\mathbf{k},\omega,\sigma} \tag{90}$$

and $C_k(\mathbf{k})^{-1} = \sum_{h=k}^0 f_h(\mathbf{k})$ and

$$\mathcal{V}(\psi) = \sum_\sigma \left[g^p_{2,0} F^{(0)}_{2,\sigma,\sigma} + g^o_{2,0} F^{(0)}_{2,\sigma,-\sigma} + g_{4,0} F^{(0)}_{3,\sigma,-\sigma}\right]. \tag{91}$$

The above model is apparently quite different with respect to the Hubbard model. It is *not* spin symmetric, but it is invariant under the separate left and right phase transformations (88), while such invariance is not verified (even asymptotically) in the Hubbard model for the presence of the backward scattering. Despite such differences, from the analysis of (90) we will get the proof of (75) for any m_2, m_3.

The multiscale integration of (89) can be performed as in for the Hubbard model, up to some minor modification; after the scales $0, -1, \ldots, h, h > k$, are integrated one finds

$$\mathrm{e}^{S^{(h+1)}(\phi,J)} \int P_{Z_{h-1},C_{h-1}}(\mathrm{d}\psi^{(\leqslant h-1)}) \int P_{Z_{h-1},\tilde{f}_h^{-1}}(\mathrm{d}\psi^{(h)})\, \mathrm{e}^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) + \hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)},\phi,J)}, \tag{92}$$

where $P_{Z_h,C_h}(\mathrm{d}\psi^{(\leqslant h)})$ is the Grassmann integration with propagator $\hat{g}^{(\leqslant k)}_\omega(\mathbf{k}) = \frac{1}{Z_h} \frac{C^{-1}_{h,k}}{-\mathrm{i}k_0+\omega k}$ and

$$\mathcal{L}\mathcal{V}^h(\psi) = \sum_\sigma \left[g^o_{2,h} F^{(h)}_{2,\sigma,-\sigma} + g^p_{2,h} F^{(h)}_{2,\sigma,\sigma} + g_{4,h} F^{(h)}_{4,\sigma,-\sigma}\right] \tag{93}$$

and $\nu_h = \delta_h = 0$ by the oddness of the propagator $\hat{g}^{(k)}_\omega(\mathbf{k}) = -\hat{g}^{(k)}_\omega(-\mathbf{k})$; moreover,

$$\mathcal{L}\mathcal{B}^{(h)}\big(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}, \phi, 0\big) = \tilde{Z}^{(2)\varepsilon}_{h-1} \frac{1}{\sqrt{2}} \sum_{\omega,\sigma} \int \mathrm{d}\mathbf{x} \phi_{\mathbf{x},\varepsilon}(\sigma)^\varepsilon \psi^+_{\omega,\sigma,\mathbf{x}} \psi^-_{\omega,\sigma,\mathbf{x}}. \tag{94}$$

By a combination of ward identities and Dyson equation, it has been proved in [8] that for any $h \leqslant 0$, there exists an ε such that, for a suitable constant C, if $|g^p_{2,0}|, |g^o_{2,0}|, |g^p_{4,0}| \leqslant \bar{\varepsilon}$:

$$|g^o_{2,h} - g^o_{2,0}|, |g^p_{2,h} - g^p_{2,0}|, |g^p_{4,h} - g_{4,0}| \leqslant C\bar{\varepsilon}^2. \tag{95}$$

By the analogous of bounds (53), it is easy to derive the following expressions for the correlations at the cut-off scale; if $\bar{\mathbf{k}}_1 = -\bar{\mathbf{k}}_2 = \bar{\mathbf{k}}$, $|\bar{\mathbf{k}}_1| = \gamma^k$ and γ^k is the infrared cut-off, if $\vec{g}_h = g^o_{2,h}, g^p_{2,h}, g_{4,h}$ and $\varepsilon_k = \sup_{h\geqslant k} |\vec{g}_h|$:

$$\begin{aligned} \langle \rho_{\varepsilon,2\bar{\mathbf{k}}}; \psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1} \psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2} \rangle_T &= \frac{Z^{\varepsilon,(2)}_k}{(Z_k)^2} \frac{1}{(D_\omega(\bar{\mathbf{k}}))^2} \left(\frac{(\sigma')^\varepsilon}{\sqrt{2}} + O(\varepsilon_k) \right); \\ \langle \psi^+_{\omega,\sigma,\bar{\mathbf{k}}_1} \psi^-_{\omega,\sigma,\bar{\mathbf{k}}_1} \rangle &= \frac{1}{Z_k} \frac{1}{D_\omega(\bar{\mathbf{k}})} (1 + O(\varepsilon_k)), \end{aligned} \tag{96}$$

where $D_\omega(\mathbf{k}) = -\mathrm{i}k_0 + \omega k$ and $\rho_{\sigma,\mathbf{p}} = \frac{1}{L\beta} \sum_{\mathbf{k}} \sum_\omega \psi^+_{\omega,\sigma,\mathbf{k+p}} \psi^+_{\omega,\sigma,\mathbf{k}}$ and $\rho_{\varepsilon,\mathbf{p}} = \frac{1}{\sqrt{2}} \sum_{\omega,\sigma} (\sigma)^\varepsilon \rho_{\sigma,\mathbf{p}}$.

If $\vec{g}_k = (g^o_{2,h}, g^p_{2,h}, g^p_{4,h})$, the effective renormalization $Z^{(2),\varepsilon}_h$ verifies

$$\frac{Z^{(2),\varepsilon}_{h-1}}{Z_{h-1}} = \frac{Z^{(2),\varepsilon}_h}{Z_h} [1 + \beta^h_2(\vec{g}_h, \vec{g}_{h+1}, \ldots, \vec{g}_0)] \tag{97}$$

and

$$\beta^h_2(\vec{g}_h, \vec{g}_{h+1}, \ldots, \vec{g}_{-1}) = \beta^h_{2,a}(\vec{g}_h, \vec{g}_h, \ldots, \vec{g}_h) + O(\bar{\varepsilon}\gamma^{\frac{h}{2}}), \tag{98}$$

with

$$\beta^h_{2,a}(\vec{g}_h, \vec{g}_h, \ldots, \vec{g}_h) = \sum_{n_1,n_2,n_3} b^h_{n_1,n_2,n_3} (g^o_{2,h})^{n_1} (g^p_{2,h})^{n_2} (g^o_{4,h})^{n_3} + O(\gamma^{\frac{h}{2}}), \tag{99}$$

with $b^h_{n_1,n_2,n_3} = b_{n_1,n_2,n_3} + O(\gamma^{\frac{h}{2}})$. In the following section, we will find a ward identity, based on the phase symmetry (88), relating $\langle \rho_{\varepsilon,2\bar{\mathbf{k}}}; \psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1} \psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2} \rangle_T$ with $\langle \psi^+_{\omega,\sigma,\bar{\mathbf{k}}_i} \psi^-_{\omega,\sigma,\bar{\mathbf{k}}_i} \rangle$, with $i = 1, 2$; from such relation we will find the identity

$$\frac{Z^{(2)\varepsilon}_k}{Z_k} = 1 + O(\varepsilon_k). \tag{100}$$

The compact support properties of the functions $f_j(\mathbf{k})$ used in the multiscale decomposition imply that $Z^{(2)\varepsilon}_k$ is essentially equal in the functional integral (89) *with* or *without* the infrared cut-off γ^k; hence it is easy to see that (100) implies

$$b_{n_1,n_2,n_3} = 0 \tag{101}$$

for any n_1, n_2, n_3; by definition, with $c_{m^o_1,m^p_1,m^o_2,m^p_2,m_3}$ defined in (71),

$$c_{0,0,m^o_2,m^p_2,m_3} = b_{m^o_2,m^p_2,m_3} = 0 \tag{102}$$

and, using (72), (74) and (75) follows.

3.2. Ward identities

It remains to prove (100), and this will be done by deriving a set of ward identities in the auxiliary model.

By performing in $\mathcal{H}^\varepsilon$ the local-phase transformation

$$\begin{aligned} &\psi^\pm_{\mathbf{x},\omega,\sigma} \to \mathrm{e}^{\pm \mathrm{i}\alpha_{\mathbf{x}}} \psi^\pm_{\mathbf{x},\omega,\sigma}, \qquad &&\psi^\pm_{\mathbf{x},-\omega,\sigma} \to \psi^\pm_{\mathbf{x},-\omega,\sigma}, \\ &\psi^\pm_{\mathbf{x},\omega,-\sigma} \to \psi^\pm_{\mathbf{x},\omega,-\sigma}, \qquad &&\psi^\pm_{\mathbf{x},-\omega,-\sigma} \to \psi^\pm_{\mathbf{x},-\omega,-\sigma}, \end{aligned} \tag{103}$$

we find

$$\mathcal{H}_\varepsilon(0,J)=\log\int P(\mathrm{d}\psi)\,\mathrm{e}^{-\int \mathrm{d}\mathbf{x}\psi^+_{\mathbf{x},\omega,\sigma}(\mathrm{e}^{\mathrm{i}\alpha_\mathbf{x}}D\,\mathrm{e}^{-\mathrm{i}\alpha_\mathbf{x}}-D)\psi^-_{\mathbf{x},\omega,\sigma}-V+\sum_{\varepsilon,\omega',\sigma'\neq\omega,\sigma}\psi^\varepsilon_{\mathbf{x},\omega',\sigma'}J^{-\varepsilon}_{\mathbf{x},\omega',\sigma'}}$$
$$\times\,\mathrm{e}^{\int \mathrm{d}\mathbf{x}(\mathrm{e}^{\mathrm{i}\alpha_\mathbf{x}}-1)\psi^+_{\mathbf{x},\omega,\sigma}J^-_{\mathbf{x},\omega,\sigma}+(\mathrm{e}^{-\mathrm{i}\alpha_\mathbf{x}}-1)J^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}},$$

where $D\psi^\pm_{\mathbf{x},\omega,\sigma}=\frac{1}{\beta L}\sum_\mathbf{k}\mathrm{e}^{\pm\mathrm{i}\mathbf{k}\mathbf{x}}C_{k,0}(\mathbf{k})(-\mathrm{i}k_0+\omega k)\psi^\pm_{\mathbf{x},\omega,\sigma}$. We can write $\int\mathrm{d}\mathbf{x}\psi^+_{\mathbf{x},\omega,\sigma}(\mathrm{e}^{\mathrm{i}\alpha_\mathbf{x}}D\,\mathrm{e}^{-\mathrm{i}\alpha_\mathbf{x}}-D)\psi^-_{\mathbf{x},\omega,\sigma}$ as

$$\int\mathrm{d}\mathbf{x}\alpha_\mathbf{x}\big[\psi^+_{\mathbf{x},\omega,\sigma}D\psi^-_{\mathbf{x},\omega,\sigma}-\big(D\psi^+_{\mathbf{x},\omega,\sigma}\big)\psi^-_{\mathbf{x},\omega,\sigma}+O(\alpha_\mathbf{x})\big].\tag{105}$$

Of course, the presence of the cut-off has the effect that $\psi^+_{\mathbf{x},\omega,\sigma}D\psi^-_{\mathbf{x},\omega,\sigma}-(D\psi^+_{\mathbf{x},\omega,\sigma})\psi^-_{\mathbf{x},\omega,\sigma}$ is not simply equal to $D(\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma})$. By making the derivative with respect to $\alpha_\mathbf{x}$, $J_\mathbf{x}$, $J_\mathbf{y}$ the following ward identities are found:

$$D_\omega(\mathbf{p})\langle\rho_{\mathbf{p},\omega,\sigma};\psi^+_{\mathbf{k}_1,\omega',\sigma'}\psi^-_{\mathbf{k}_2,\omega',\sigma'}\rangle_T=\delta_{\omega,\omega'}\delta_{\sigma,\sigma'}\big[\langle\psi^+_{\mathbf{k}_1,\omega',\sigma'}\psi^-_{\mathbf{k}_1,\omega',\sigma'}\rangle$$
$$-\langle\psi^+_{\mathbf{k}_2,\omega',\sigma'}\psi^-_{\mathbf{k}_2,\omega',\sigma'}\rangle\big]+\langle\delta\rho_{\omega,\sigma,\mathbf{p}};\psi^+_{\mathbf{k}_1\omega',\sigma'}\psi^-_{\mathbf{k}_2,\omega',\sigma'}\rangle,\tag{106}$$

where

$$\delta\rho_{\omega,\sigma,\mathbf{p}}=\frac{1}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}}C_\omega(\mathbf{k},\mathbf{p})\psi^+_{\mathbf{k}+\mathbf{p},\omega,\sigma}\psi^-_{\mathbf{k},\omega,\sigma}.\tag{107}$$

and

$$C_\omega(\mathbf{k}^+,\mathbf{k}^-)=(C_{k,0}(\mathbf{k}^-)-1)D_\omega(\mathbf{k}^-)-(C_{k,0}(\mathbf{k}^+)-1)D_\omega(\mathbf{k}^+)$$

with respect to the formal ward identities valid in a theory in the absence of cut-offs, we have in (106) the presence of a *correction* term (the last one) due to the fact that the cut-offs break the invariance under local-phase transformation.

Remark. The validity of (106) can be checked at lowest orders from the following trivial identity:

$$g^{[k,0]}_\omega(\mathbf{k})g^{[k,0]}_\omega(\mathbf{k}+\mathbf{p})=\frac{g^{[k,0]}_\omega(\mathbf{k})-g^{[k,0]}_\omega(\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})}-g^{[k,0]}_\omega(\mathbf{k})g^{[k,0]}_\omega(\mathbf{k}+\mathbf{p})\frac{C_\omega(\mathbf{k},\mathbf{k}+\mathbf{p})}{D_\omega(\mathbf{p})}\tag{108}$$

replacing the well-known identity valid in the absence of cut-off

$$\frac{D_\omega(\mathbf{p})}{D_\omega(\mathbf{k})D_\omega(\mathbf{k}+\mathbf{p})}=\frac{1}{D_\omega(\mathbf{k})}-\frac{1}{D_\omega(\mathbf{k}+\mathbf{p})}.\tag{109}$$

Neglecting the last addend in the rhs of (106), choosing $\mathbf{k}_1=-\mathbf{k}_2$, $|\mathbf{k}_1|=\gamma^k$ and using (96) one immediately gets (100); however the presence of the last term (which is not smaller than the others) could prevent to derive such relation.

We prove now the following crucial *correction identity*:

$$\langle\delta\rho_{\mathbf{p},\omega,\sigma};\psi^+_{\mathbf{k},\omega',\sigma'}\psi^-_{\mathbf{k}+\mathbf{p},\omega',\sigma'}\rangle=H_{\omega,\sigma,\omega',\sigma'}(\mathbf{k},\mathbf{p})$$
$$+\sum_{\omega'',\sigma''}\big[\nu^1_{\omega,\sigma;\omega'',\sigma''}p+i\nu^0_{\omega,\sigma;\omega'',\sigma''}p_0\big]\langle\rho_{\mathbf{p},\omega'',\sigma''};\psi^+_{\mathbf{k},\omega',\sigma'}\psi^-_{\mathbf{k}+\mathbf{p},\omega',\sigma'}\rangle,\tag{110}$$

where $\nu^1_{\omega,\sigma;\omega'',\sigma''}$, $\nu^0_{\omega,\sigma;\omega'',\sigma''}$ are real and such that $|\nu_{\omega,\sigma;\omega'',\sigma''}|\leqslant C\bar{\varepsilon}$ and, if $\bar{\mathbf{k}}_1=-\bar{\mathbf{k}}_2$, $|\bar{\mathbf{k}}_1|=\gamma^k$, $\bar{\mathbf{p}}=\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2$:

$$|H_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{k}}_1,\bar{\mathbf{p}})|\leqslant C\bar{\varepsilon}\frac{\gamma^{-k}}{Z_k}.\tag{111}$$

Remark. In the spinless case, an analogous equation holds, see [3], with the identity $\nu^1_{\omega;\omega''} = \nu^0_{\omega;\omega''}$ due to symmetry reasons.

In order to prove (110), we write $H_{\omega,\sigma,\omega',\sigma'}$ in (110) as a functional integral, $\mathbf{p} = \mathbf{k}_1 - \mathbf{k}_2$:

$$H_{\omega,\sigma,\omega',\sigma'}(\mathbf{k}_1, \mathbf{p}) = \frac{\partial^3}{\partial \phi_{\mathbf{p},\omega,\sigma}\, \partial J^-_{\mathbf{k}_1,\omega',\sigma'}\, \partial J^+_{\mathbf{k}_2,\omega',\sigma'}} \tilde{\mathcal{W}}(\phi, J)|_{J=\phi=0}, \tag{112}$$

where

$$\tilde{\mathcal{W}}(\phi, J) = \log \int P(\mathrm{d}\psi)\, \mathrm{e}^{-V - \sum_{\varepsilon,\omega,\sigma} \psi^\varepsilon_{\mathbf{x},\omega,\sigma} J^{-\varepsilon}_{\mathbf{x},\omega,\sigma}}\, \mathrm{e}^{\bar{T}_{0,\omega,\sigma} - \sum_{\omega',\sigma'} [\nu^1_{\omega,\sigma;\omega',\sigma'} p + \mathrm{i} \nu^0_{\omega,\sigma;\omega',\sigma'} p_0] \bar{T}_{\omega,\sigma;\omega',\sigma'}}, \tag{113}$$

with

$$\begin{aligned} \bar{T}_{0,\omega,\sigma}(\psi) &= \frac{1}{(\beta L)^2} \sum_{\mathbf{k},\mathbf{p}} \phi^\varepsilon_{\mathbf{p},\omega} C_\omega(\mathbf{k}, \mathbf{k}-\mathbf{p}) \psi^+_{\mathbf{k},\omega,\sigma} \psi^-_{\mathbf{k}-\mathbf{p},\omega,\sigma} \equiv \frac{1}{\beta L} \sum_{\mathbf{p}\neq 0} \phi(\mathbf{p}) \delta\rho_{\mathbf{p},\omega,\sigma}, \\ \bar{T}_{\omega,\sigma;\omega',\sigma'}(\psi) &= \frac{1}{(\beta L)^2} \sum_{\mathbf{k},\mathbf{p}} J_{\mathbf{p},\omega,\sigma} \psi^+_{\mathbf{k},\omega',\sigma'} \psi^-_{\mathbf{k}-\mathbf{p},\omega',\sigma'}. \end{aligned} \tag{114}$$

We perform again a multiscale analysis for (113) (for more details in the spinless case, see [3]). We shall use some properties of the operator $C_\omega(\mathbf{k}, \mathbf{k}-\mathbf{p})$. Let us consider first the effect of contracting both $\hat{\psi}$ fields of $\delta\rho$ on the same or two different scales; hence, we have to study the quantity

$$\Delta^{(i,j)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = g^{(i)}(\mathbf{k}^+) g^{(j)}(\mathbf{k}^-) C_\omega(\mathbf{k}^+, \mathbf{k}^-) = 0 \qquad k < i, \quad j < 0, \tag{115}$$

where $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$. The crucial observation is that

$$\Delta^{(i,j)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = 0, \qquad \text{if} \quad k < i, \quad j < 0, \tag{116}$$

since $C^{-1}_{k,0}(\mathbf{k}^\pm) = 1$, if $k < i$, $j < 0$. Let us then consider the cases in which $\Delta^{(i,j)}_\omega(\mathbf{k}^+, \mathbf{k}^-)$ is not identically equal to 0. Since $\Delta^{(i,j)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = \Delta^{(j,i)}_\omega(\mathbf{k}^-, \mathbf{k}^+)$, we can restrict the analysis to the case $i \geqslant j$. We define $u_0(\mathbf{k}) = 0$ if $|\mathbf{k}| \leqslant 1$ and $u_0(\mathbf{k}) = 1 - f_0(\mathbf{k})$ if $1 \leqslant |\mathbf{k}|$. Moreover, we define $u_k(\mathbf{k}) = 0$ if $|\mathbf{k}| \geqslant \gamma^k$ and $u_k(\mathbf{k}) = 1 - f_k(\mathbf{k})$ if $|\mathbf{k}| \leqslant \gamma^k$.

Then we get, for $|\mathbf{p}| \geqslant \gamma^h$,

$$\Delta^{(0,0)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = \left[\frac{f_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)} u_0(\mathbf{k}^-) - \frac{f_0(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)} u_0(\mathbf{k}^+) \right], \tag{117}$$

$$\Delta^{(k,k)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{\tilde{Z}_{k-1}(\mathbf{k}^+) \tilde{Z}_{k-1}(\mathbf{k}^-)} \left[\frac{f_k(\mathbf{k}^+) u_k(\mathbf{k}^-)}{D_\omega(\mathbf{k}^+)} - \frac{u_k(\mathbf{k}^+) f_k(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)} \right], \tag{118}$$

$$\Delta^{(0,k)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{\tilde{Z}_{k-1}(\mathbf{k}^-)} \left[\frac{f_0(\mathbf{k}^+) u_k(\mathbf{k}^-)}{D_\omega(\mathbf{k}^+)} - \frac{f_k(\mathbf{k}^-) u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^-)} \right], \tag{119}$$

$$\Delta^{(0,j)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = -\frac{1}{Z_{j-1}} \frac{\tilde{f}_j(\mathbf{k}^-) u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^-)}, \qquad k < j < 0, \tag{120}$$

$$\Delta^{(i,k)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{\tilde{Z}_{k-1}(\mathbf{k}^-) Z_{i-1}} \frac{\tilde{f}_i(\mathbf{k}^+) u_k(\mathbf{k}^-)}{D_\omega(\mathbf{k}^+)}, \qquad j = k < i \leqslant -1. \tag{121}$$

As an easy consequence of the above equations, one can write, for $0 \geqslant j > k$,

$$\Delta^{(0,j)}_\omega(\mathbf{k}^+, \mathbf{k}^-) = \mathbf{p} \mathbf{S}^{(j)}_\omega(\mathbf{k}^+, \mathbf{k}^-), \tag{122}$$

where $S^{(j)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)$ are smooth functions such that

$$\left|\partial^{m_+}_{\mathbf{k}^+}\partial^{m_j}_{\mathbf{k}^-}S^{(j)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)\right| \leqslant C_{m_0+m_j}\frac{\gamma^{-j(1+m_j)}}{Z_{j-1}}. \tag{123}$$

Finally, it is easy to see that, if $0 > i \geqslant k$,

$$\left|\Delta^{(i,k)}_{\omega}(\mathbf{k}^+,\mathbf{k}^-)\right| \leqslant C|\mathbf{p}|\gamma^{-(i-k)}\frac{\gamma^{-k-i}}{Z_{i-1}}. \tag{124}$$

Note that, in the rhs of (124), there is apparently a Z^{-1}_{k-1} factor missing, which is a consequence of the fact that $\tilde{Z}_{k-1}(\mathbf{k}) = 1$ for $|\mathbf{k}| \leqslant \gamma^{k-1}$. Note also the presence in the bound of the extra factor $\gamma^{-(i-k)}$, with respect to the dimensional bound; it will allow us to avoid renormalization of the marginal terms containing $\Delta^{(i,k)}$. After the integration of ψ^0 we get an expression like (37), and the terms linear in J and quadratic in ψ in the exponent will be denoted by $K^{(-1)}_{\phi}$; we write $K^{(-1)}_{\phi} = K^{(a,-1)}_{\phi} + K^{(b,-1)}_{\phi}$, where $K^{(a,-1)}_{\phi}$ was obtained by the integration of $\bar{T}_0$ and $K^{(b,-1)}_{\phi}$ from the integration of $\bar{T}$. We can write $K^{(a,-1)}_{\phi}$ as

$$K^{(a,-1)}_{\phi} = \sum_{\omega}\int \mathrm{d}\mathbf{x}\phi_{\mathbf{x},\omega,\sigma}\Big\{\bar{T}_{0,\omega,\sigma} + \sum_{\tilde{\omega},\tilde{\sigma}}\int \mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\big[F^{(-1)}_{2,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}(\mathbf{x},\mathbf{y},\mathbf{z}) + F^{(-1)}_{1,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}(\mathbf{x},\mathbf{y},\mathbf{z})\delta_{\omega,\tilde{\omega}}\delta_{\sigma,\tilde{\sigma}}\big]\big[\psi^+_{\mathbf{y},\tilde{\omega},\tilde{\sigma}}\psi^-_{\mathbf{z},\tilde{\omega},\tilde{\sigma}}\big]\Big\}, \tag{125}$$

where $F^{(-1)}_{2,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}$ and $F^{(-1)}_{1,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}$ represent the terms in which both or only one of the fields in $\delta\rho_{\mathbf{p},\omega,\sigma}$, respectively, are contracted. Both contributions to the rhs of (125) are dimensionally marginal; however, the regularization of $F^{(-1)}_{1,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}$ is trivial, as it is of the form

$$F^{(-1)}_{1,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}(\mathbf{k}^+,\mathbf{k}^-) = [[C_{k,0}(\mathbf{k}^-) - 1]D_{\omega}(\mathbf{k}^-)\hat{g}^{(0)}_{\omega}(\mathbf{k}^+) - u_0(\mathbf{k}^+)]G^{(2)}_{\omega}(\mathbf{k}^+) \tag{126}$$

or the similar one, obtained exchanging $\mathbf{k}^+$ with $\mathbf{k}^-$. By the oddness of the propagator in the momentum, $G^{(2)}_{\omega}(0) = 0$, hence we can regularize such term without introducing any local term, by simply rewriting it as

$$F^{(-1)}_{1,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}(\mathbf{k}^+,\mathbf{k}^-) = [G^{(2)}_{\omega}(\mathbf{k}^+) - G^{(2)}_{\omega}(0)][(C_{k,0}(\mathbf{k}^-) - 1)D_{\omega}(\mathbf{k}^-)\hat{g}^{(0)}_{\omega}(\mathbf{k}^+) - u_0(\mathbf{k}^+)], \tag{127}$$

$F^{(-1)}_{2,\omega,\sigma;\tilde{\omega},\tilde{\sigma}}$ can be written as

$$F^{(-1)}_{2,\omega,\sigma;\tilde{\omega},\tilde{\sigma}} = \frac{1}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}}\phi_{\mathbf{p},\omega,\sigma}[p_0\mathrm{i}W_{0;\omega,\sigma,\omega',\sigma'}(\mathbf{k},\mathbf{k}+\mathbf{p}) + pW_{1;\omega,\sigma,\omega',\sigma'}(\mathbf{k},\mathbf{k}+\mathbf{p})]\psi^+_{\mathbf{k},\omega',\sigma'}\psi^-_{\mathbf{k}+\mathbf{p},\omega',\sigma'}, \tag{128}$$

and we define the localization as

$$\mathcal{L}F^{(-1)}_{2,\omega,\sigma;\tilde{\omega},\tilde{\sigma}} = \frac{1}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}}[\mathrm{i}p_0W_{0;\omega,\sigma,\omega',\sigma'}(0,0) + pW_{1;\omega,\sigma,\omega',\sigma'}(0,0)]\psi^+_{\mathbf{k},\omega',\sigma'}\psi^-_{\mathbf{k}+\mathbf{p},\omega',\sigma'}. \tag{129}$$

Note that $W_{0;\omega,\sigma,\omega',\sigma'}(0,0)$ and $W_{1;\omega,\sigma,\omega',\sigma'}(0,0)$ are real. As a consequence of the above definition

$$\mathcal{L}K^{(-1)}_{\phi} = \bar{T}_{0,\omega,\sigma}(\psi^{(\leqslant -1)}) + \sum_{\omega',\sigma'}\big[\nu^1_{-1;\omega,\sigma;\omega',\sigma'}p + \mathrm{i}\nu^0_{-1;\omega,\sigma;\omega',\sigma'}p_0\big]\bar{T}_{\omega,\sigma;\omega',\sigma'}(\psi^{(\leqslant -1)}). \tag{130}$$

The above integration procedure can be iterated with no important differences up to scale $k+1$. In particular, for all the marginal terms such that one of the fields in $\bar{T}_{0,\omega,\sigma}$ is contracted at scale j, we put $\mathcal{R} = 1$; in fact the second field has to be contracted at scale k and, by (121), the extra factor γ^{k-j} has the effect of automatically regularizing such contributions.

$\nu^\alpha_{j-1;\omega,\sigma;\omega',\sigma'}$ verify, for $k+1 \leqslant j \leqslant -1$, $\alpha = 0, 1$, the following recursive equation:

$$\nu^\alpha_{j-1;\omega,\sigma;\omega',\sigma'} = \nu^\alpha_{j;\omega,\sigma;\omega',\sigma'} + \beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \nu_j \ldots, \vec{g}_0, \nu_0), \tag{131}$$

with

$$\begin{aligned}\beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \nu_j \ldots, \vec{g}_0, \nu_0) &= \beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \ldots, \vec{g}_0) \\ &+ \sum_{\alpha'=0,1} \sum_{\omega'',\sigma''} \sum_{j'=j}^{0} \nu^{\alpha'}_{j';\omega,\sigma;\omega'',\sigma''} \beta^{\alpha';j,j'}_{\omega',\sigma';\omega'',\sigma''}(\vec{g}_j, \ldots, \vec{g}_0)\end{aligned} \tag{132}$$

and

$$\left|\beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \ldots, \vec{g}_0)\right| \leqslant C\varepsilon\gamma^{\frac{j}{2}}, \qquad \left|\beta^{\alpha;j,j'}_{\omega',\sigma';\omega'',\sigma''}(\vec{g}_j, \ldots, \vec{g}_0)\right| \leqslant C\bar{\varepsilon}^2\gamma^{-\frac{1}{2}|j-j'|}. \tag{133}$$

It is possible to find $\nu^\alpha_{\omega,\sigma;\omega',\sigma'}$, so that

$$\left|\nu^\alpha_{j,\omega,\sigma;\omega',\sigma'}\right| \leqslant c_0\varepsilon_k\gamma^{\frac{j}{4}}. \tag{134}$$

This is done by choosing

$$\nu^\alpha_{\omega,\sigma;\omega',\sigma'} = -\sum_{j=k}^{0} \beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \nu_j \ldots, \vec{g}_0, \nu_0), \tag{135}$$

which implies

$$\nu^\alpha_{i;\omega,\sigma;\omega',\sigma'} = -\sum_{j\leqslant i} \beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \nu_j \ldots, \vec{g}_0, \nu_0). \tag{136}$$

We consider the Banach space $\mathcal{M}$ of sequences $\underline{\nu}_{j,\omega,\sigma;\omega',\sigma'} = \{\nu_{j,\omega,\sigma;\omega',\sigma'}\}_{j\leqslant 0}$ such that, for any α, ω, σ, for a constant c, $\left\|\underline{\nu}^\alpha_{j,\omega,\sigma;\omega',\sigma'}\right\| = \sup_{j\leqslant 0}\gamma^{\frac{-j}{4}}\left|\nu^\alpha_{j,\omega,\sigma;\omega',\sigma'}\right| \leqslant c$. For any $\underline{\nu}, \underline{\nu}' \in \mathcal{M}$,

$$|\vec{g}_j(\underline{\nu}) - \vec{g}_0| \leqslant C\varepsilon_k^2 \qquad |\vec{g}_j(\underline{\nu}) - \vec{g}_j(\underline{\nu}')| \leqslant C\varepsilon_k\gamma^{\frac{j}{4}}\|\underline{\nu} - \underline{\nu}'\|. \tag{137}$$

We look for a fixed point of the operator $\mathbf{T} : \mathcal{M} \to \mathcal{M}$ defined as

$$\mathbf{T}(\underline{\nu})_{j,\omega,\sigma;\omega',\sigma'} = \sum_{j\leqslant i} \beta^\alpha_{j;\omega,\sigma;\omega',\sigma'}(\vec{g}_j, \nu_j .., \vec{g}_0, \nu_0). \tag{138}$$

The operator $\mathbf{T} : \mathcal{M} \to \mathcal{M}$ is a contraction. In fact $\mathbf{T}$ leaves $\mathcal{M}$ invariant as

$$|\mathbf{T}(\underline{\nu})_j| \leqslant \sum_{i\leqslant j} C\varepsilon_k\gamma^{\frac{i}{4}} \leqslant C_1\varepsilon\gamma^{\frac{j}{4}}, \tag{139}$$

and $|\mathbf{T}(\underline{\nu})_j - \mathbf{T}(\underline{\nu}')_j| \leqslant C\varepsilon_k\gamma^{\frac{j}{4}}\|\nu - \nu'\|$.

Note finally that, from an explicit computation of (131), we get, if $i = 1, 0$,

$$\begin{aligned} &\nu_{\omega,\sigma;\omega,\sigma} = O\left(\varepsilon_k^2\right); &&\nu_{\omega,\sigma;-\omega,-\sigma} = -ag_{2,o} + O\left(\varepsilon_k^2\right) \\ &\nu_{\omega,\sigma;-\omega,\sigma} = -ag_{2,p} + O\left(\varepsilon_k^2\right); &&\nu_{\omega,\sigma;\omega,\sigma} = -ag_{4,o} + O\left(\varepsilon_k^2\right), \end{aligned} \tag{140}$$

where a is a suitable coefficient. In fact the lowest order contribution to $\nu_{\omega,\sigma;\omega',\sigma'}$ can be obtained from the graph represented in figure 3, whose local part is (in the limit $L, \beta \to \infty$)

$$\mathcal{L}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} C_\omega(\mathbf{k}, \mathbf{k}-\mathbf{p})\hat{g}^{(\leqslant 0)}_\omega(\mathbf{k})\hat{g}^{(\leqslant 0)}_\omega(\mathbf{k}-\mathbf{p}), \tag{141}$$

which is equal to, up to terms $O(|\gamma - 1|)$

$$-D_{-\omega}(\mathbf{p})\frac{1}{2}\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{\chi_0'(|\mathbf{k}|)}{|\mathbf{k}|} = -D_{-\omega}(\mathbf{p})\frac{1}{4\pi}\int_1^\infty \mathrm{d}\rho\,\chi_0'(\rho) = \frac{D_{-\omega}(\mathbf{p})}{4\pi}. \tag{142}$$

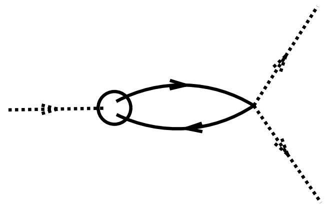

Figure 3. Lowest order contribution to $\nu_{\omega,\sigma;\omega',\sigma'}$; the small circle represents C_ω.

To $H_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{k}}_1, \bar{\mathbf{p}})$ contribute then two kind of terms; the first kind of terms are obtained by the contraction of terms $\sum_{\omega',\sigma'}[\nu^1_{i;\omega,\sigma;\omega',\sigma'}p + \mathrm{i}\nu^0_{i;\omega,\sigma;\omega',\sigma'}p_0]\bar{T}_{\omega,\sigma;\omega',\sigma'}$ at a certain scale i, while the second kind of terms are obtained by the contraction of terms $\bar{T}_{0,\omega,\sigma}(\psi^{(\leqslant -1)})$. The first kind of terms are bounded by the dimensional bound $\frac{\gamma^{-2k}}{Z_k^2}$ (see (53)) times a factor $\gamma^{\frac{k-i}{2}}$ (for the short memory property), times a factor $\gamma^{\frac{i}{4}}$ (from (134)); the final bound for such terms is then

$$C\varepsilon|\bar{\mathbf{p}}|\frac{\gamma^{-2k}}{Z_k^2}\gamma^{\frac{k}{4}}. \tag{143}$$

The second kind of terms can have one of the two fields in $\delta\rho$ contracted at scale 0; such terms again admit the bound $C\varepsilon|\bar{\mathbf{p}}|\frac{\gamma^{-2k}}{Z_k^2}\gamma^{\frac{k}{4}}$, for the short memory property. The last possibility is that one of the two fields in $\delta\rho$ is contracted at scale k; in such a case we get the bound $C\varepsilon|\bar{\mathbf{p}}|\frac{\gamma^{-2k}}{Z_k}$; note that there are no contributions of this kind of order 0 in ε. This concludes the proof of (111).

By combining (110) and (106) we obtain, if $\bar{\mathbf{k}}_1 = -\bar{\mathbf{k}}_2$, $|\bar{\mathbf{k}}_1| = \gamma^k$, $\bar{\mathbf{p}} = 2\bar{\mathbf{k}}_1$:

$$\begin{aligned}
\big(-\mathrm{i}\bar{p}_0\big(1 - \nu^0_{\omega,\sigma;\omega,\sigma}\big) &+ \omega\bar{p}(1 - \nu^1_{\omega,\sigma;\omega,\sigma})\big\langle\rho_{\omega,\sigma,\bar{\mathbf{p}}};\, \psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2}\big\rangle_T \\
&= \delta_{\omega,\omega'}\delta_{\sigma,\sigma'}\big[\big\langle\psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_1}\big\rangle - \big\langle\psi^+_{\omega',\sigma',\bar{\mathbf{k}}_2}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2}\big\rangle\big] \\
&\quad + \sum_{\omega'',\sigma''\neq\omega,\sigma}\big[\nu^1_{\omega,\sigma;\omega'',\sigma''}\bar{p} + \mathrm{i}\nu^0_{\omega,\sigma;\omega'',\sigma''}\bar{p}_0\big]\big\langle\rho_{\omega'',\sigma'',\bar{\mathbf{p}}};\, \psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2}\big\rangle_T \\
&\quad + H_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{k}}_1, \bar{\mathbf{p}}),
\end{aligned} \tag{144}$$

and after same algebra we find

$$\begin{aligned}
\big\langle\rho_{\omega,\sigma,\bar{\mathbf{p}}};\, \psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2}\big\rangle_T &= R_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{k}}_1, \bar{\mathbf{p}}) + \big[\delta_{\omega,\omega'}\delta_{\sigma,\sigma'}a_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}}) \\
&\quad + \delta_{-\omega,\omega'}\delta_{\sigma,\sigma'}b_{-\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}}) + \delta_{\omega,\omega'}\delta_{-\sigma,\sigma'}c_{\omega,-\sigma,\omega',\sigma'}(\bar{\mathbf{p}}) \\
&\quad + \delta_{-\omega,\omega'}\delta_{-\sigma,\sigma'}d_{-\omega,-\sigma,\omega',\sigma'}(\bar{\mathbf{p}})\big]\big[\big\langle\psi^+_{\omega',\sigma',\bar{\mathbf{k}}_1}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2}\big\rangle - \big\langle\psi^+_{\omega',\sigma',\bar{\mathbf{k}}_2}\psi^-_{\omega',\sigma',\bar{\mathbf{k}}_2}\big\rangle\big]
\end{aligned} \tag{145}$$

and $a_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}}) = [D_\omega(\bar{\mathbf{p}})]^{-1}(1 + O(\varepsilon_k))$,

$$\begin{aligned}
&\big|b_{-\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}})|, |c_{\omega,-\sigma,\omega',\sigma'}(\bar{\mathbf{p}})|, |d_{-\omega,-\sigma,\omega',\sigma'}(\bar{\mathbf{p}})| \leqslant C\varepsilon_k\gamma^{-k}; \\
&|R_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}})| \leqslant C\varepsilon_k\frac{\gamma^{-2k}}{Z_k}.
\end{aligned} \tag{146}$$

Summing over σ, ω and using (96), we get

$$\begin{aligned}
\frac{Z_k^{\varepsilon,(2)}}{Z_k}\frac{\gamma^{-2k}}{Z_k}\left(\frac{(\sigma')^\varepsilon}{\sqrt{2}} + O(\varepsilon_k)\right) &= \sum_{\sigma,\omega}(\sigma)^\varepsilon R_{\omega,\sigma,\omega',\sigma'} + \sum_{\sigma,\omega}(\sigma)^\varepsilon[\delta_{\omega,\omega'}\delta_{\sigma,\sigma'}a_{\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}}) \\
&\quad + \delta_{-\omega,\omega'}\delta_{\sigma,\sigma'}b_{-\omega,\sigma,\omega',\sigma'}(\bar{\mathbf{p}}) + \delta_{\omega,\omega'}\delta_{-\sigma,\sigma'}c_{\omega,-\sigma,\omega',\sigma'}(\bar{\mathbf{p}})
\end{aligned}$$

$$+\delta_{-\omega,\omega'}\delta_{-\sigma,\sigma'}d_{-\omega,-\sigma,\omega',\sigma'}(\bar{\mathbf{p}})]\frac{\gamma^{-k}}{Z_k}\left(\frac{1}{\sqrt{2}}+O(\varepsilon_k)\right) \tag{147}$$

from which (100) follows.

4. Conclusions

We have proved the absence of logarithmic corrections in the spin and charge density correlations of the repulsive Hubbard model at zero momentum; this proves that the divergences at zero momentum, found in [7], are an artefact of their perturbative approach.

One cannot simply follow the strategy used to prove the analogous statement for the spinless Hubbard model in [3]; while in the spinless case the Hubbard model is asymptotic to a model which is invariant under separate chiral phase transformations, in the spinning case it is asymptotic to a model which is not invariant under such transformations, at fixed spin. To solve this problem we have used an auxiliary model which is not spin symmetric but it is invariant under separate chiral and spin phase transformations, and we have used information from such model to prove a set of cancellations in the Hubbard model, finally implying the absence of logarithmic divergences.

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